☆ CMB

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Boltzmann eq. (GR version)

$$\frac{Df}{d\lambda} = C[f]$$

$$p^\mu = \frac{Dx^\mu}{d\lambda} = (p^0, p^i) = p^\mu \left(p \equiv \sqrt{g_{ij} p^i p^j}, \hat{n}^i\right)$$

$$\frac{Df}{d\lambda} = \frac{Dx^\mu}{d\lambda}\frac{\partial f}{\partial x^\mu} + \frac{Dp}{d\lambda}\frac{\partial f}{\partial p} + \frac{D\hat{n}^i}{d\lambda}\frac{\partial f}{\partial \hat{n}^i}$$

$f = f(x^\mu, p^\mu)$, $p = p(t)$

$$= p^0 \frac{\partial f}{\partial t} + p^i \frac{\partial f}{\partial x^i} + \underbrace{\frac{Dx^0}{d\lambda}}_{\because\, p = p(t)} \frac{dp}{dx^0}\frac{\partial f}{\partial p} + \underbrace{\frac{D\hat{n}^i}{d\lambda}\frac{\partial f}{\partial \hat{n}^i}}_{\text{at least 2nd order}}$$

$$\Rightarrow \frac{\partial f}{\partial t} + \frac{p^i}{p^0}\frac{\partial f}{\partial x^i} + \frac{dp}{dt}\frac{\partial f}{\partial t} = \frac{C[f]}{p^0}$$

☆ photon : $m=0,\ \mathcal{E}=p$

$\boxed{1}$ $\dfrac{p^i}{p^0}$

photon : $P^2 = g_{\mu\nu}P^\mu P^\nu = g_{00}(P^0)^2 + \underbrace{g_{ij}P^iP^j}_{\equiv p^2} = -(1+2\Psi)(P^0)^2 + p^2 = 0$

$ds^2 = -(1+2\Psi)dt^2 + a^2(1+2\Phi)\delta_{ij}dx^i dx^j$ (conformal Newtonian gauge)

$\therefore P^0 = (1+2\Psi)^{-\frac{1}{2}}p = (1-\Psi)p \quad \rightarrow \quad p = (1+\Psi)P^0$

$P^i = C\hat{p}^i \qquad \delta_{ij}\hat{p}^i\hat{p}^j = 1$

$p^2 = g_{ij}P^iP^j = a^2(1+2\Phi)\delta_{ij}\hat{p}^i\hat{p}^jC^2 = a^2(1+2\Phi)C^2$

$\therefore C = (1+2\Phi)^{-\frac{1}{2}}\dfrac{p}{a} = (1-\Phi)\dfrac{p}{a}$

$= (1-\Phi)\dfrac{p}{a}\hat{p}^i$

$\therefore \dfrac{P^i}{P^0} = \dfrac{(1-\Phi)p\hat{p}^i/a}{(1-\Psi)p} = \underbrace{\dfrac{\hat{p}^i}{a}}_{\text{1st}}\underbrace{(1+\Psi-\Phi)}_{\text{1st}} \simeq \dfrac{\hat{p}^i}{a}$

$\boxed{2}$ $\dfrac{dp}{dt} = \dfrac{Dp}{dt} = \dfrac{D}{dt}\left[(1+\Psi)P^0\right] = \dfrac{d\Psi}{dt}P^0 + (1+\Psi)\dfrac{DP^0}{dt}$

$\dfrac{d\Psi}{dt} = \dfrac{\partial\Psi}{\partial t} + \dfrac{dx^i}{dt}\dfrac{\partial\Psi}{\partial x^i}$ $\qquad \dfrac{DP^0}{dt} = \dfrac{DP^0}{d\lambda}\dfrac{d\lambda}{dt} = -\Gamma^0_{\alpha\beta}P^\alpha P^\beta\dfrac{1}{P^0}$

$\dfrac{dx^i}{dt} = \dfrac{dx^i}{d\lambda}\Big/\dfrac{dt}{d\lambda} = P^i/P^0 \simeq \dfrac{\hat{p}^i}{a}$

$= \left(\dot{\Psi} + \Psi_{,i}\dfrac{\hat{p}^i}{a}\right)(1-\cancel{\Psi})p - (1+\Psi)\Gamma^0_{\mu\nu}\dfrac{P^\mu P^\nu}{P^0}$

$\Gamma^0_{\mu\nu}\dfrac{P^\mu P^\nu}{P^0} = \Gamma^0_{00}P^0 + 2\Gamma^0_{0i}P^i + \Gamma^0_{ij}\dfrac{P^iP^j}{P^0}$

$= \dot{\Psi}P^0 + 2\Psi_{,i}P^i + \left\{a^2H\delta_{ij} - a^2\left[-\dot{\Phi} + 2H(\Psi-\Phi)\right]\delta_{ij}\right\}\dfrac{P^iP^j}{P^0}$

$$\simeq \dot{\Psi} p + 2\Psi_{,i}\, p\, \frac{\hat{p}^i}{a} + \cancel{a}\,\underline{\delta_{ij}}\left[H+\dot{\Phi}-2H(\Psi-\Phi)\right](1-2\Phi)\frac{p^2}{\cancel{a^2}}\underbrace{\hat{p}^i\hat{p}^j}_{=1}\;\frac{1+\Psi}{\cancel{p}}$$

$$= \dot{\Psi} p + 2\Psi_{,i}\, p\, \frac{\hat{p}^i}{a} + p\underbrace{\left[H+\dot{\Phi}-2H(\Psi-\Phi)\right](1-2\Phi+\Psi)}_{\simeq H + H\Psi - \cancel{2H\Phi} + \dot{\Phi} - 2H\Psi + \cancel{2H\Phi} = H + \dot{\Phi} - H\Psi}$$

$$= p\left(\dot{\Psi} + 2\Psi_{,i}\frac{\hat{p}^i}{a} + H + \dot{\Phi} - H\Psi\right)$$

$$= p\left[\dot{\Psi} + \Psi_{,i}\frac{\hat{p}^i}{a} - \underbrace{(1+\Psi)\left(\dot{\Psi} + 2\Psi_{,i}\frac{\hat{p}^i}{a} + H + \dot{\Phi} - H\Psi\right)}_{\simeq \dot{\Psi} + 2\Psi_{,i}\frac{\hat{p}^i}{a} + H + \dot{\Phi} - \cancel{H\Psi} + \cancel{H\Psi}}\right]$$

$$= p\left(-H - \dot{\Phi} - \Psi_{,i}\frac{\hat{p}^i}{a}\right)$$

$$\therefore\ \frac{\partial f}{\partial t} + \frac{p^i}{p^0}\frac{\partial f}{\partial x^i} + \frac{dp}{dt}\frac{\partial f}{\partial t}$$

$$= \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} - p\left(H + \dot{\Phi} + \Psi_{,i}\frac{\hat{p}^i}{a}\right)\frac{\partial f}{\partial p} = \frac{C[f]}{p^0}$$

$$f = \frac{1}{e^{p/T}-1} = \frac{1}{e^{p/T(1+\Theta)}-1} = \frac{1}{e^{p/T}-1} + \frac{e^{p/T}}{(e^{p/T}-1)^2}\frac{p}{T}\Theta + \cdots$$

$$\Theta \equiv \frac{\delta T}{T}$$

$$\equiv f_0 + \delta f$$

$$\frac{\partial f_0}{\partial p} = \frac{-e^{p/T}}{(\quad)^2}\frac{1}{T} = -\frac{T}{p}\frac{\partial f_0}{\partial T}$$

$$\frac{\partial f_0}{\partial T} = \frac{-e^{p/T}}{(\quad)^2}\cdot\frac{-p}{T^2}$$

<u>0th order</u>

$$\frac{\partial f_0}{\partial t} - pH\frac{\partial f_0}{\partial p} = \frac{\partial f_0}{\partial T}\frac{\partial T}{\partial t} - pH\frac{\partial f_0}{\partial p}$$

$$= \left(-\frac{p}{T}\frac{\partial T}{\partial t} - pH\right)\frac{\partial f_0}{\partial p} = 0$$

$$\therefore\ \frac{\dot{T}}{T} = -\frac{\dot{a}}{a} \qquad \therefore\ T \propto \frac{1}{a}$$

//

1st order

$\delta f = -p \frac{\partial f_0}{\partial p} \Theta$

$$\frac{\partial \delta f}{\partial t} + \frac{\hat{p}^i}{a} \underbrace{\frac{e^{p/T}}{(e^{p/T}-1)^2} \frac{p}{T}}_{= -p\frac{\partial f_0}{\partial p}} \frac{\partial \Theta}{\partial x^i} - p\left(H + \dot{\Phi} + \Psi_{,i} \frac{\hat{p}^i}{a}\right) \frac{\partial}{\partial p}\left(f_0 - p\frac{\partial f_0}{\partial p}\Theta\right)$$

$$= -p\left(\frac{\partial f_0}{\partial p}\frac{\partial \Theta}{\partial t} + \frac{\partial T}{\partial t}\frac{\partial^2 f_0}{\partial T \partial p}\Theta\right) - p\frac{\partial f_0}{\partial p}\frac{\hat{p}^i}{a}\frac{\partial \Theta}{\partial x^i} + Hp\frac{\partial}{\partial p}\left(p\frac{\partial f_0}{\partial p}\right)\Theta$$

$$- p\left(\dot{\Phi} + \Psi_{,i}\frac{\hat{p}^i}{a}\right)\frac{\partial f_0}{\partial p}$$

$\because$ fixed by T

$$\frac{\partial^2 f_0}{\partial T \partial p} = \frac{\partial}{\partial p}\left(\frac{\partial f_0}{\partial T}\right) = \frac{\partial}{\partial p}\left(-\frac{p}{T}\frac{\partial f_0}{\partial p}\right)$$

$$= -p\left[\frac{\partial f_0}{\partial p}\dot{\Theta} - \frac{\dot{T}}{T}\frac{\partial}{\partial p}\left(p\frac{\partial f_0}{\partial p}\right)\Theta\right] - p\frac{\partial f_0}{\partial p}\frac{\hat{p}^i}{a}\frac{\partial \Theta}{\partial x^i} + Hp\frac{\partial}{\partial p}\left(p\frac{\partial f_0}{\partial p}\right)\Theta$$

$$- p\left(\dot{\Phi} + \Psi_{,i}\frac{\hat{p}^i}{a}\right)\frac{\partial f_0}{\partial p}$$

$$= p\Theta\frac{\partial}{\partial p}\left(p\frac{\partial f_0}{\partial p}\right)\left(H + \frac{\dot{T}}{T}\right) = 0$$

$$= -p\frac{\partial f_0}{\partial p}\left(\dot{\Theta} + \frac{\hat{p}^i}{a}\Theta_{,i} + \dot{\Phi} + \frac{\hat{p}^i}{a}\Psi_{,i}\right) = 0$$

$\boxed{3}$ $\frac{C[f]}{p}$; $a + b + \cdots \leftrightarrow i + j + \cdots$

$$C[f] = \int d\Pi_a d\Pi_b \cdots d\Pi_i d\Pi_j \cdots \qquad d\Pi = \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}$$

$$\times (2\pi)^4 \delta^{(4)}(p_a + p_b + \cdots - p_i - p_j - \cdots)$$

$$\times \left[|\mathcal{M}|^2_{a+b+\cdots \to i+j+\cdots} f_a f_b \cdots (1 \pm f_i)(1 \pm f_j)\cdots\right.$$

$$\left. - |\mathcal{M}|^2_{i+j+\cdots \to a+b+\cdots} f_i f_j \cdots (1 \pm f_a)(1 \pm f_b)\cdots\right]$$

$e^-(q) + \gamma(p) \leftrightarrow e^-(q') + \gamma(p')$

i) $|\mathcal{M}|^2_\rightarrow = |\mathcal{M}|^2_\leftarrow$: reversible

ii) $1 \pm f_i \simeq 1$; BE / FD → MB

$$C[f(p)] \simeq \int \frac{d^3q}{(2\pi)^3 2E_q} \frac{d^3q'}{(2\pi)^3 2E_{q'}} \frac{d^3p'}{(2\pi)^3 2E_{p'}} (2\pi)^4 \underbrace{\delta^{(4)}(p+q-p'-q')}_{\delta^{(3)}(\vec{p}+\vec{q}-\vec{p}'-\vec{q}')\,\delta(E_\gamma(p)+E_e(q)-E_\gamma(p')-E_e(q'))} |\mathcal{M}|^2$$

$$\times \left[f_e(q') f(p') - f_e(q) f(p)\right]$$

$\swarrow\ q' \times$

$E_\gamma(p) = p,\quad E_e(q) = m_e + \frac{q^2}{2m_e} \simeq m_e$

$$\simeq \int \frac{d^3q}{(2\pi)^3 2m_e} \frac{d^3p'}{(2\pi)^3 2p'} \frac{(2\pi)^4}{(2\pi)^3 2m_e} \delta\left(p + \frac{q^2}{2m_e} - p' - \frac{|\vec{p}+\vec{q}-\vec{p}'|^2}{2m_e}\right) |\mathcal{M}|^2$$

$$\times \left[f_e(\vec{p}+\vec{q}-\vec{p}') f(p') - f_e(q) f(p)\right]$$

$$E_e(q) - E_e(|\vec{p}+\vec{q}-\vec{p}'|) = \frac{q^2}{2m_e} - \frac{|\vec{p}-\vec{p}'+\vec{q}|^2}{2m_e}$$

$p' \simeq p$

(NR Compton scattering)

→ elastic

$$\simeq \frac{\cancel{q^2} - \left[\cancel{q^2} + 2\vec{q}\cdot(\vec{p}-\vec{p}')\right]}{2m_e} = \frac{(\vec{p}'-\vec{p})\cdot\vec{q}}{m_e}$$

$$\delta(p - p' + E_e(q) - E_e(|\vec{p}+\vec{q}-\vec{p}'|))$$

$f(x) = f(x_0) + (x-x_0) f'(x_0) + \cdots$

$$\simeq \delta(p-p') + \left[E_e(q) - E_e(|\cdots|)\right] \left.\frac{\partial\, \delta(p-p'+X)}{\partial X}\right|_{X=0}$$

$$= \delta(p-p') + \frac{(\vec{p}'-\vec{p})\cdot\vec{q}}{m_e} \cdot -\frac{\partial}{\partial(p-p')} \delta(p-p')$$

$$= \delta(p-p') + \frac{(\vec{p}'-\vec{p})\cdot\vec{q}}{m_e} \frac{\partial}{\partial p'} \delta(p-p')$$

$\frac{\vec{q}}{m_e} \simeq \vec{v}_b$

$$\frac{\partial}{\partial x} f(x-y) = -\frac{\partial}{\partial y} f(x-y)$$

$$\frac{\partial}{\partial x} f(x-y) = \lim_{k\to 0} \frac{f(x+k-y) - f(x-y)}{k}$$

$$-\frac{\partial}{\partial y} f(x-y) = \lim_{-k\to 0} \frac{f(x-(y-k)) - f(x-y)}{-k} = -\lim_{-k\to 0} \frac{f(x+k-y) - f(x-y)}{k}$$

$f_e(\vec{q}+\vec{p}-\vec{p}') \simeq f_e(\vec{q})$

$$\simeq \frac{\pi}{4m_e^2}\underbrace{\int\frac{d^3q}{(2\pi)^3}}_{=n_e}\frac{d^3p'}{(2\pi)^3p'}\left[\delta(p-p')+(\vec{p}-\vec{p}\,')\cdot\vec{v}_b\frac{\partial}{\partial p'}\delta(p-p')\right]\underbrace{|\mathcal{M}|^2}_{=8\pi\sigma_T m_e^2} f_e(q)\left[f(\vec{p}\,')-f(\vec{p})\right]$$

$$= \frac{\pi}{4m_e^2}n_e\, 8\pi\sigma_T m_e^2 \underbrace{\int\frac{d^3p'}{(2\pi)^3p'}}_{=\frac{1}{(2\pi)^3}\int dp'p'\int d\Omega'}\left[\delta(p-p')+(\vec{p}-\vec{p}\,')\cdot\vec{v}_b\frac{\partial}{\partial p'}\delta(p-p')\right]$$

$$\times\left[f_0(\vec{p}\,')-p'\frac{\partial f_0}{\partial p'}\Theta(\hat{p}')-f_0(\vec{p})+p\frac{\partial f_0}{\partial p}\Theta(\hat{p})\right]$$

$$= \frac{n_e\sigma_T}{4\pi}\int dp'\,p'\,d\Omega'$$

$$\simeq \delta(p-p')\left[\cancel{f_0}(\vec{p}\,')-p'\frac{\partial f_0}{\partial p'}\Theta(\hat{p}')-\cancel{f_0}(\vec{p})+p\frac{\partial f_0}{\partial p}\Theta(\hat{p})\right]$$

$$+(\vec{p}-\vec{p}\,')\cdot\vec{v}_b\frac{\partial}{\partial p'}\delta(p-p')\left[f_0(\vec{p}\,')-f_0(\vec{p})\right]$$

$$\int_{-1}^{1}p v_b\mu\, d\mu=0$$

$$= n_e\sigma_T\int dp'p'\left\{\delta(p-p')\left[-p'\frac{\partial f_0}{\partial p'}\frac{1}{4\pi}\int d\Omega'\,\Theta(\hat{p}')+p\frac{\partial f_0}{\partial p}\Theta(\hat{p})\right]\right.$$

$$\left.+\ \vec{p}\cdot\vec{v}_b\frac{\partial}{\partial p'}\delta(p-p')\left[f_0(\vec{p}\,')-f_0(\vec{p})\right]\right\}$$

$$\Theta_\ell\equiv\frac{1}{(-i)^\ell}\int_{-1}^{1}\frac{d\mu}{2}P_\ell(\mu)\Theta(\mu)\quad\longleftrightarrow\quad\Theta=\sum_{\ell=0}^{\infty}(-i)^\ell(2\ell+1)\Theta_\ell P_\ell,\quad\int_{-1}^{1}\frac{d\mu}{2}P_\ell P_{\ell'}=\frac{\delta_{\ell\ell'}}{2\ell+1}$$

$P_0=1,\ P_1=x,\ \dots$

$$= n_e\sigma_T\left\{\int dp'p'\,\delta(p-p')\left[-p'\frac{\partial f_0}{\partial p'}\Theta_0+p\frac{\partial f_0}{\partial p}\Theta(\hat{p})\right]\right.$$

$$\left.+\ \vec{p}\cdot\vec{v}_b\int dp'p'\left[f_0(\vec{p}\,')-f_0(\vec{p})\right]\frac{\partial}{\partial p'}\delta(p-p')\right\}$$

$f'=\delta'(p-p')\qquad f=\delta(p-p')$

$g=p'\left[f_0(p')-f_0(p)\right]\qquad g'=f_0(p')-f_0(p)+p'\frac{\partial f_0}{\partial p'}$

$$= n_e\sigma_T\left[p\left(-p\frac{\partial f_0}{\partial p}\Theta_0 + p\frac{\partial f_0}{\partial p}\Theta\right) - \vec{p}\cdot\vec{v}_b\, p\frac{\partial f_0}{\partial p}\right]$$

$$= -n_e\sigma_T\, p\left(\Theta_0 - \Theta + \hat{p}\cdot\vec{v}_b\right) p\frac{\partial f_0}{\partial p} = C[f(p)]$$

$$\therefore\ \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} - p\left(H + \dot{\Phi} + \Psi_{,i}\frac{\hat{p}^i}{a}\right)\frac{\partial f}{\partial p} = \frac{C[f]}{p^0}$$

$$= -p\frac{\partial f_0}{\partial p}\left(\dot{\Theta} + \frac{\hat{p}^i}{a}\Theta_{,i} + \dot{\Phi} + \frac{\hat{p}^i}{a}\Psi_{,i}\right)$$

$$= -n_e\sigma_T\,\cancel{p}\left(\Theta_0 - \Theta + \hat{p}\cdot\vec{v}_b\right) p\frac{\partial f_0}{\partial p}\,\frac{1+\Phi}{\cancel{p}}$$

$$\simeq -n_e\sigma_T\left(\Theta_0 - \Theta + \hat{p}\cdot\vec{v}_b\right) p\frac{\partial f_0}{\partial p}$$

$$\therefore\ \dot{\Theta} + \frac{\hat{p}^i}{a}\Theta_{,i} + \dot{\Phi} + \frac{\hat{p}^i}{a}\Psi_{,i} = n_e\sigma_T\left(\Theta_0 - \Theta + \hat{p}\cdot\vec{v}_b\right)$$ //

conformal time : $\frac{d}{dt} = \frac{1}{a}\frac{d}{d\eta}$

$$\Theta' + \hat{p}^i\Theta_{,i} + \Phi' + \hat{p}^i\Psi_{,i} = \underbrace{n_e\sigma_T a}_{= -\tau',\ \text{optical depth}}\left(\Theta_0 - \Theta + \hat{p}\cdot\vec{v}_b\right)$$

$$F = \int\frac{d^3k}{(2\pi)^3}e^{ik\cdot x}F_k\ ,\quad F_{,i} = \int\frac{d^3k}{(2\pi)^3}e^{ik\cdot x}(ik_i)F_k\ ,\quad \hat{p}^iF_{,i} = \int i\underbrace{k_i\hat{p}^i}_{=|\vec{k}|\mu}F_k$$

$$\Theta_k' + ik\mu\Theta_k + \Phi_k' + ik\mu\Psi_k = -\tau'\left(\Theta_0 - \Theta_k + \mu v_b\right)$$ //

☆ CDM : $\epsilon = \sqrt{m^2+p^2}$, $P = (p^0, p^i) = (\epsilon, p^i)$

$$g_{\mu\nu} p^\mu p^\nu = g_{00}(p^0)^2 + g_{ij} p^i p^j = -(p^0)^2 + p^2 = -\epsilon^2 + p^2 = -m^2$$

$$\Rightarrow \frac{Df}{d\lambda} = \frac{Dx^\mu}{d\lambda}\frac{\partial f}{\partial x^\mu} + \frac{D\epsilon}{d\lambda}\frac{\partial f}{\partial \epsilon} + \frac{D\hat{n}^i}{d\lambda}\frac{\partial f}{\partial \hat{n}^i}$$

$$= p^0 \frac{\partial f}{\partial t} + p^i \frac{\partial f}{\partial x^i} + \frac{Dx^0}{d\lambda}\frac{d\epsilon}{dx^0}\frac{\partial f}{\partial \epsilon} + \cancel{\frac{D\hat{n}^i}{d\lambda}\frac{\partial f}{\partial \hat{n}^i}} = 0$$

($\because$ DM → collision X)

$$\therefore \frac{\partial f}{\partial t} + \frac{p^i}{p^0}\frac{\partial f}{\partial x^i} + \frac{d\epsilon}{dt}\frac{\partial f}{\partial \epsilon} = 0$$

$\boxed{1}$ $\frac{p^i}{p^0}$

$$g_{\mu\nu} p^\mu p^\nu = -(1+2\Psi)(p^0)^2 + p^2 = -m^2$$

$$\therefore (1+2\Psi)^{1/2} p^0 = \sqrt{m^2+p^2} = \epsilon \qquad \therefore p^0 = (1-\Psi)\epsilon$$

$p^i = (1-\Phi)\, p \frac{\hat{p}^i}{a}$ (the same)

$$\therefore \frac{p^i}{p^0} = \frac{(1-\Phi)\, p\hat{p}^i/a}{(1-\Psi)\epsilon} = \frac{\hat{p}^i}{a}\frac{p}{\epsilon}(1+\Psi-\Phi) \simeq \frac{\hat{p}^i}{a}\frac{p}{\epsilon}$$

$\boxed{2}$
$$\frac{d\epsilon}{dt} = \frac{D\epsilon}{dt} = \frac{D}{dt}\left[(1+\Psi)p^0\right] = \left(\dot{\Psi} + \frac{\hat{p}^i}{a}\frac{p}{\epsilon}\Psi_{,i}\right)p^0 - (1+\Psi)\Gamma^0_{\mu\nu}\frac{p^\mu p^\nu}{p^0}$$

$$\Gamma^0_{\mu\nu}\frac{p^\mu p^\nu}{p^0} = \dot{\Psi}p^0 + 2\Psi_{,i}p^i + \cancel{a^2}\underline{\underline{\delta_{ij}}}\left(H + \dot{\Phi} - 2H\Psi + 2H\Phi\right)(1-2\Phi)\frac{p^2}{\cancel{a^2}}\underbrace{\hat{p}^i\hat{p}^j}_{=1}\frac{1+\Phi}{\epsilon}$$

$$\simeq \dot{\Psi}\epsilon + 2\Psi_{,i}p\frac{\hat{p}^i}{a} + \frac{p^2}{\epsilon}\left(H + \dot{\Phi} - H\Psi\right)$$

$$\simeq \left(\cancel{\dot{\Psi}} + \cancel{\frac{\hat{p}^i}{a}\frac{p}{\epsilon}\Psi_{,i}}\right)\epsilon - \left[\cancel{\dot{\Psi}\epsilon} + \cancel{2}\Psi_{,i}p\frac{\hat{p}^i}{a} + \frac{p^2}{\epsilon}\left(H + \dot{\Phi} - \cancel{H\Psi}\right) + \cancel{\frac{p^2}{\epsilon}H\Psi}\right]$$

$$= -\left(H\frac{p^2}{\epsilon} + \frac{p^2}{\epsilon}\dot{\Phi} + p\Psi_{,i}\frac{\hat{p}^i}{a}\right)$$

$$\therefore\ \frac{\partial f}{\partial t} + \frac{p}{\epsilon}\frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} - \left(H\frac{p^2}{\epsilon} + \frac{p^2}{\epsilon}\dot{\varphi} + p\Psi_{,i}\frac{\hat{p}^i}{a}\right)\frac{\partial f}{\partial \epsilon} = 0$$

$\boxed{1}$ continuity : $n = \int \frac{d^3p}{(2\pi)^3} f$, $v^i = \frac{1}{n}\int \frac{d^3p}{(2\pi)^3} f \frac{p\hat{p}^i}{\epsilon}$

$$\times \int \frac{d^3p}{(2\pi)^3}$$

$$\frac{\partial}{\partial t}\int \frac{d^3p}{(2\pi)^3} f + \frac{1}{a}\frac{\partial}{\partial x^i}\int \frac{d^3p}{(2\pi)^3} f\frac{p\hat{p}^i}{\epsilon} - (H+\dot{\varphi})\int \frac{d^3p}{(2\pi)^3}\frac{\partial f}{\partial \epsilon}\frac{p^2}{\epsilon}$$

$$- \frac{1}{a}\underbrace{\Psi_{,i}}_{\text{1st order}}\underbrace{\int \frac{d^3p}{(2\pi)^3}\frac{\partial f}{\partial \epsilon}p\hat{p}^i}_{\text{non-zero for } \delta f \text{ : 1st-order}} = 0 \qquad = \frac{\partial p}{\partial \epsilon}\frac{\partial f}{\partial p}\frac{p^2}{\epsilon} = \frac{\cancel{\epsilon}}{p}\frac{\partial f}{\partial p}\frac{p^{\cancel{2}}}{\cancel{\epsilon}}$$

$\times$

$$\frac{\partial}{\partial t}n + \frac{1}{a}\frac{\partial}{\partial x^i}(nv^i) - (H+\dot{\varphi})\int \frac{d^3p}{(2\pi)^3}p\frac{\partial f}{\partial p} = 0$$

$$= \frac{4\pi}{(2\pi)^3}\int dp\, p^3\frac{\partial f}{\partial p} = -3\frac{4\pi}{(2\pi)^3}\int dp\, p^2 f = -3\int\frac{d^3p}{(2\pi)^3}f$$

$f' = f'$ $f = f$

$g = p^3$ $g' = 3p^2$

$= -3n$

$$\therefore\ \frac{\partial n}{\partial t} + \frac{1}{a}\frac{\partial}{\partial x^i}(nv^i) + 3(H+\dot{\varphi})n = 0$$

0-th order

$$\frac{\partial n_0}{\partial t} + 3Hn_0 = 0 \longrightarrow n \propto \frac{1}{a^3}\,_{//}$$

$$\frac{\delta n}{n_0} = \frac{\delta(mn)}{mn_0} = \frac{\delta\rho}{\rho_0} = \delta$$

1st order

$$\frac{\partial \delta n}{\partial t} + \frac{n_0}{a}\frac{\partial v^i}{\partial x^i} + 3\underbrace{(H+\dot{\varphi})(n_0+\delta n)}_{= \underline{Hn_0} + H\delta n + \dot{\varphi}n_0} = 0 \quad \leftarrow \times \frac{1}{n_0}, \quad -\frac{\delta n}{n_0}\frac{\dot{n}_0}{n_0} + \frac{\dot{\delta n}}{n_0}\underbrace{\frac{\dot{n}_0}{n_0}}_{=-3H}$$

$$\therefore\ \frac{\partial \delta}{\partial t} + \frac{1}{a}\frac{\partial v^i}{\partial x^i} + 3\dot{\varphi} = 0\,_{//}$$

$\boxed{2}$ Euler : $\times \int \frac{d^3p}{(2\pi)^3} \frac{p\hat{p}^j}{E}$

$\not\text{A}$ $p^\mu, x^\mu \to$ indep.

$\frac{p}{E} \sim v$; 2nd order, X

$$\frac{\partial}{\partial t}\int \frac{d^3p}{(2\pi)^3} f \frac{p\hat{p}^j}{E} + \frac{1}{a}\frac{\partial}{\partial x^i}\int \frac{d^3p}{(2\pi)^3} f \frac{p^2\hat{p}^i\hat{p}^j}{E^2}$$

$$-(H+\dot{\Phi})\int \frac{d^3p}{(2\pi)^3} \underbrace{\frac{\partial f}{\partial E}}_{=\frac{E}{p}\frac{\partial f}{\partial p}} \frac{p^2}{E}\frac{p\hat{p}^j}{E} - \frac{\Psi_{,i}}{a}\int \frac{d^3p}{(2\pi)^3} \underbrace{\frac{\partial f}{\partial E}}_{=\frac{E}{p}\frac{\partial f}{\partial p}} \frac{p^2}{E}\hat{p}^i\hat{p}^j = 0$$

$$\frac{\partial}{\partial t}(nv^j) - (H+\dot{\Phi})\underbrace{\int \frac{d^3p}{(2\pi)^3}\frac{\partial f}{\partial p}\frac{p^2}{E}\hat{p}^j}_{} - \frac{\Psi_{,i}}{a}\underbrace{\int \frac{d^3p}{(2\pi)^3}\frac{\partial f}{\partial p}\, p\,\hat{p}^i\hat{p}^j}_{} = 0$$

$$= \int \frac{d\Omega}{(2\pi)^3}\hat{p}^j \int dp\, \frac{p^4}{E}\frac{\partial f}{\partial p}$$

$f' = f'$ $\quad f = f$

$g = \frac{p^4}{E}$ $\quad g' = 4\frac{p^3}{E} - \frac{p^4}{E}\cdot\frac{p}{E}$ $\quad \times$

$$= -4\int dp\, p^2 f \frac{p}{E}$$

$$= -4\int \frac{d^3p}{(2\pi)^3} f \frac{p}{E}\hat{p}^j = -4nv^j$$

$$= \int \frac{d\Omega}{(2\pi)^3}\hat{p}^i\hat{p}^j \int dp\, p^3 \frac{\partial f}{\partial p}$$

$$= \frac{1}{(2\pi)^3}\frac{4\pi}{3}\delta^{ij}$$

$f' = f'$ $\quad f = f$

$g = p^3$ $\quad g' = 3p^2$

$$= \frac{\delta^{ij}}{3}\int \frac{d\Omega}{(2\pi)^3} \qquad = -3\int dp\, p^2 f$$

$$= -\delta^{ij}\int \frac{d^3p}{(2\pi)^3} f = -\delta^{ij} n$$

$$\therefore \underbrace{\frac{\partial}{\partial t}(nv^j)} + 4(H + \underbrace{\dot{\Phi}}_{\times})nv^j + \frac{\Psi_{,i}}{a}\delta^{ij} n = 0$$

$$= n\frac{\partial v^j}{\partial t} + v^j \cdot -3Hn$$

$$\therefore \frac{\partial v^i}{\partial t} + Hv^i + \frac{\Psi^{,i}}{a} = 0$$

conformal time : $\delta' + v^i_{;i} + 3\Phi' = 0$, $v^{i\prime} + aHv^i + \Psi^{,i} = 0$

$\not\text{A}$ vel. divergence : $\vec{\nabla}\cdot\vec{v} \equiv \theta$, irrotational vel : $v^i = v^{,i}$, $\theta = \Delta v$

$$\delta_k' + \theta_k + 3\Phi_k' = 0$$

$$\theta_k' + aH\theta_k - k^2\Psi_k = 0$$

☆ baryon : e+p

$e(q)+\gamma(p) \leftrightarrow e(q')+\gamma(p')$, $p(Q)+e(q) \leftrightarrow p(Q')+e(q')$

$$\frac{\partial f_e}{\partial t}+\cdots = \frac{1}{p^0}\left[\langle C_{ep}\rangle_{QQ'q'}+\langle C_{e\gamma}\rangle_{pp'q'}\right] , \quad p^0=E_e \simeq m_e$$

$$\frac{\partial f_p}{\partial t}+\cdots = \frac{1}{p^0}\langle C_{ep}\rangle_{Q'qq'} , \quad p^0=E_p \simeq m_p$$

$\boxed{1}$ continuity : $\times \int \frac{d^3q(Q)}{(2\pi)^3}$

RHS = 0 ∵ n-conserving process

$\therefore\ \delta_k' + \theta_k + 3\Phi_k' = 0$ (the same as DM)

$\boxed{2}$ Euler : $\times \int \frac{d^3q}{(2\pi)^3}\frac{q}{E_e}q^i + \int \frac{d^3Q}{(2\pi)^3}\frac{Q}{E_p}Q^i$

$\times m_e$ $\times m_p$ $m_p \gg m_e$

$$\underbrace{m_p\frac{\partial}{\partial t}(nv^j)}_{=n\frac{\partial v^j}{\partial t}+v^j\cdot -3Hn} + 4Hm_p nv^j + \frac{\Phi_{,i}}{a}\delta^{ij}m_p n = \underbrace{\langle C_{ep}(q^j+Q^j)\rangle_{QQ'qq'}}_{=0\ \because\ \text{mom. cons.}} + \underbrace{\langle C_{e\gamma}q^j\rangle_{pp'qq'}}_{=-\langle C_{e\gamma}p^j\rangle_{pp'qq'}}$$

$= \rho_b\left(\dot v^j + Hv^j + \frac{1}{a}\Phi^{,j}\right)$, $m_p n = \rho_b$

$$\langle C_{e\gamma}p^j\rangle_{pp'qq'} = \int \frac{d^3p}{(2\pi)^3}\frac{p^j}{2p}\underbrace{C[f(p)]}_{=-n_e\sigma_T p\left(\Theta_0 - \Theta + \hat p\cdot\vec v_b\right)p\frac{\partial f_0}{\partial p}}$$

☆ See e.g. Ma & Bertschinger ...

$$\to\ \theta_b' + aH\theta_b - k^2\Phi_k = -\frac{4\rho_\gamma}{3\rho_b}\tau'(\theta_\gamma - \theta_b)$$

☆ large scale anisotropy

$\Theta_1 = i\int_{-1}^{1}\frac{d\mu}{2}\mu\Theta$

$\Theta_1 = \int_{-1}^{1}\frac{d\mu}{2}\Theta$

① photon : $\Theta_k' + ik\mu\Theta_k + \Phi_k' + ik\mu\Psi_k = -\tau'(\Theta_0 - \Theta_k + \mu v_b)$

$\times \underset{=1}{P_0},\ \int_{-1}^{1}\frac{d\mu}{2} \longrightarrow \Theta_0' + ik\underbrace{\int_{-1}^{1}\frac{d\mu}{2}\mu\Theta}_{=-i\Theta_1} + \Phi_k' = 0$

$\therefore$ monopole eq. : $\Theta_0' + k\Theta_1 = -\Phi_k'$

$\times \underset{=\mu}{P_1},\ \int_{-1}^{1}\frac{d\mu}{2} \longrightarrow \underbrace{\int_{-1}^{1}\frac{d\mu}{2}\mu\Theta_k'}_{=-i\Theta_1'} + ik\int_{-1}^{1}\frac{d\mu}{2}\mu^2\Theta_k + ik\int_{-1}^{1}\frac{d\mu}{2}\mu^2\Psi_k$

$= \tau'\underbrace{\int_{-1}^{1}\frac{d\mu}{2}\mu\Theta_k}_{=-i\Theta_1} - \tau' v_b\int_{-1}^{1}\frac{d\mu}{2}\mu^2$

$= \int_{-1}^{1}\frac{d\mu}{2}\left[\frac{2}{3}\left(\frac{3}{2}\mu^2 - \frac{1}{2}\right) + \frac{1}{3}\right]\Theta_k$

$= \int_{-1}^{1}\frac{d\mu}{2}P_2\Theta_k + \frac{1}{3}\int_{-1}^{1}\frac{d\mu}{2}\Theta_k = (\dots)\Theta_2 + \frac{\Theta_0}{3}$

$\therefore -i\Theta_1' + ik\frac{\Theta_0}{3} + ik\frac{\Psi_k}{3} = -i\tau'\left(\Theta_1 - i\frac{v_b}{3}\right)$

$\therefore$ dipole eq. : $\Theta_1' - \frac{k}{3}\Theta_0 - \frac{k}{3}\Psi_k = \tau'\left(\Theta_1 - i\frac{v_b}{3}\right)$

② CDM ; $\begin{cases} \delta_k' + \theta_k + 3\Phi_k' = 0 \\ \theta_k' + aH\theta_k - k^2\Psi_k = 0 \end{cases}$

$\theta = v^i{}_{,i} \rightarrow \theta_k = ik_i v^i_k = ik_i\hat{k}^i v_k$
$= ikv_k$

$\longrightarrow \begin{cases} \delta_k' + ikv_k + 3\Phi_k' = 0 \\ v_k' + aHv_k + ik\Psi_k = 0 \end{cases}$

③ baryon $\begin{cases} \delta_k' + ikv_k + 3\Phi_k' = 0 \\ v_b' + aHv_b + ik\Psi_k = \frac{\tau'}{R}(v_b + 3i\Theta_1) \end{cases}$

$k \to 0$: from monopole eq. $\Theta_0' = -\Phi_k'$

$\therefore \Theta_0 = \underset{\sim\sim}{\Theta_0^{(ini)}} - \Phi_k$

$\Theta_0(\eta_0) = \Theta_0^{(ini)} - \Phi(0) = \frac{\Phi(0)}{2}$ $\quad\therefore \Theta_0^{(ini)} = \frac{3\Phi(0)}{2} \equiv \frac{3\Phi_p}{2}$

$\therefore \Theta_0 = \underset{\sim\sim}{-\Phi_k} + \frac{3}{2}\Phi_p = \frac{3}{5}\Phi_p = \frac{3}{5}\cdot\frac{10}{9}\Phi = \frac{2}{3}\Phi$

$\xrightarrow{MP} = \frac{9}{10}\Phi_p$

temperature we see today ; $\Theta_0 + \Psi$

$\uparrow$ $\frac{\delta T}{T}$ $\quad\uparrow$ grav. pot. $\because$ photons had to travel out of the initial potentials

$\therefore (\Theta_0 + \Psi)_* \simeq (\Theta_0 - \Phi)_* \simeq -\frac{1}{3}\Phi_* \simeq \frac{1}{3}\Psi_*$

from CDM eq. $\delta_k = \underset{\sim\sim}{\delta_k^{(ini)}} - 3\Phi_k = 2\Phi$

$\delta_k(\eta_0) = \frac{3}{2}\Phi_p$, $\therefore \delta_k^{(ini)} = \frac{9}{2}\Phi_p = \frac{9}{2}\cdot\frac{10}{9}\Phi = 5\Phi$

$\therefore$ in terms of δ_k, $(\Theta_0 + \Psi)_* \simeq -\frac{\delta_k}{6}$ //

(SW)

☆ tight coupling : $\tau \gg 1$

$$v_b' + aHv_b + ik\Psi_k = \frac{\tau'}{R}(v_b + 3i\Theta_1)$$

$$\rightarrow v_b = -3i\Theta_1 + \frac{R}{\tau'}(v_b' + aHv_b + ik\Psi_k) \qquad R \equiv \frac{3\rho_b}{4\rho_\gamma}$$

$\gg$

$$\simeq -3i\Theta_1 + \frac{R}{\tau'}\left(-3i\Theta_1' - 3i\,aH\Theta_1 + ik\Psi_k\right)$$

$$\Theta_1' - \frac{k}{3}\Theta_0 - \frac{k}{3}\Psi_k = \tau'\left(\Theta_1 - i\frac{v_b}{3}\right)$$

$$\simeq \tau'\left\{\Theta_1 - \frac{i}{3}\cdot -3i\left[\Theta_1 + \frac{R}{\tau'}\left(\Theta_1' + aH\Theta_1 - \frac{k}{3}\Psi_k\right)\right]\right\}$$

$$= -R\Theta_1' - RaH\Theta_1 + \frac{k}{3}R\Psi_k$$

$$(1+R)\Theta_1' + RaH\Theta_1 - \frac{k}{3}\Theta_0 - \frac{k}{3}(1+R)\Psi_k = 0$$

$$\therefore \Theta_1' + aH\frac{R}{1+R}\Theta_1 - \frac{k}{3(1+R)}\Theta_0 = \frac{k}{3}\Psi_k$$

taking 1 deriv. of monopole eq.

$$\Theta_0'' + k\Theta_1' = -\Phi_k''$$

$$= \Theta_0'' + k\left[\frac{k}{3}\Psi_k - aH\frac{R}{1+R}\Theta_1 + \frac{k}{3(1+R)}\Theta_0\right] \qquad \leftarrow k\Theta_1 = -\Theta_0' - \Phi_k'$$

$$= \Theta_0'' + aH\frac{R}{1+R}\Theta_0' + \frac{k^2}{3(1+R)}\Theta_0 + \frac{k^2}{3}\Psi_k + aH\frac{R}{1+R}\Phi_k' \qquad c_s \equiv \frac{1}{\sqrt{3(1+R)}}$$

$$\therefore \Theta_0'' + aH\frac{R}{1+R}\Theta_0' + c_s^2k^2\Theta_0 = -\frac{k^2}{3}\Psi_k - aH\frac{R}{1+R}\Phi_k' - \Phi_k''$$

$$R' = a'\frac{dR}{da} = a'\frac{3}{4}\frac{\rho_\gamma\rho_b' - \rho_b\rho_\gamma'}{\rho_\gamma^2} = \frac{3}{4}\frac{\rho_b}{\rho_\gamma}a'\left(\frac{\rho_b'}{\rho_b} - \frac{\rho_\gamma'}{\rho_\gamma}\right) \qquad \rho_b \sim a^{-3},\ \rho_\gamma \sim a^{-4}$$

$$= a'(-3a^{-1} + 4a^{-1})R = aHR \,//$$

$$\therefore \ \Theta_0'' + \frac{R'}{1+R}\Theta_0' + c_s^2 k_2 \Theta = -\Phi_k'' - \frac{R'}{1+R}\Phi_k' - c_s^2 k^2 \Phi_k + c_s^2 k_2 \Phi_k - \frac{k^2}{3}\Psi_k$$

$$c_s^2 = \frac{1}{3(1+R)}$$

$$\therefore \left(\frac{d^2}{d\eta^2} + aH\frac{R}{1+R}\frac{d}{d\eta} + c_s^2 k^2\right)\Theta_0 + \Phi_k = -\frac{k^2}{3}\Psi_k + \frac{k^2}{3(1+R)}\Phi_k$$

$$= \frac{k^2}{3}\left(\frac{\Phi_k}{1+R} - \Psi_k\right)$$

① homogeneous eq.

$$\left(\frac{d^2}{d\eta^2} + \underbrace{\frac{R'}{1+R}\frac{d}{d\eta}}_{\text{damping}} + \underbrace{c_s^2 k^2}_{\text{pressure}}\right)(\Theta_0 + \Phi_k) = 0$$

$$\sim \frac{R}{\eta^2}(\Theta_0 + \Phi_k) \ll \sim k c_s^2 (\Theta_0 + \Phi_k)$$

for sub-horizon

$$\therefore \ \Theta_0 + \Phi_k = \begin{cases} s[k r_s(\eta)] \equiv S_1 \\ c[k r_s(\eta)] \equiv S_2 \end{cases}$$

$r_s = \int_0^\eta c_s(\eta')\,d\eta'$, sound horizon

(comoving dist. traveled by a sound wave of speed c_s by η)

② full sol.

$$(\Theta_0 + \Phi_k)(\eta) = C_1 S_1(\eta) + C_2 S_2(\eta) + \frac{k^2}{3}\int d\eta' \left(\frac{\Phi_k}{1+R} - \Psi_k\right)\frac{S_1(\eta')S_2(\eta) - S_1(\eta)S_2(\eta')}{S_1(\eta')S_2'(\eta') - S_1'(\eta')S_2(\eta')}$$

a) adiabatic : $\Theta_0(0) = \frac{1}{2}\Phi_k(0)$, $r_s(0) = 0$

$\therefore \ C_1 = 0, \ C_2 = \Theta_0(0) + \Phi_k(0)$ (cosine wave)

(if iso $\rightarrow C_2 = 0$)

b) denominator

$$S_1 S_2' - S_1' S_2 = -kc_s\, s^2(kr_s) - kc_s\, c^2(kr_s)$$

$$= -kc_s \simeq -\frac{k}{\sqrt{3}} \qquad (R \ll 1)$$

c) numerator

$$S_1(\eta')S_2(\eta) - S_1(\eta)S_2(\eta') = s(kr_s(\eta'))\,c(\eta) - s(\eta)\,c(\eta')$$

$$= -s\left[k(r_s(\eta) - r_s(\eta'))\right]$$

$$\therefore\ (\Theta_0 + \Phi_k)(\eta) = (\Theta_0(0) + \Phi_k(0))\, c\,(kr_s)$$

$$+ \frac{k}{\sqrt{3}}\int d\eta' \left(\Phi_k - \Psi_k\right) s\left[k(r_s(\eta) - r_s(\eta'))\right]$$

"acoustic oscillation"