Phenomenology of Neutrino mass matrices and the Flavor symmetries

Shivani Gupta

Yonsei University

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NEUTRINO

- Earlier twenties and thirties only weak process known was β decay of nuclei.
- Chadwick discovered that energy spectrum of electrons from β decay (transition of nucleus (A,Z) to (A,Z+1)) is continuous which cannot be if conservation of energy and momentum is true.
- 1) In β decay energy is not conserved.
  2) Total energy is shared by electron and neutral penetrating particle.
- Pauli proposed "neutron" with spin **1/2**, large penetration length later renamed by Fermi as NEUTRINO (neutral and small(Italian)).
- First proof of existence of neutrinos in fifties by Cowan and Reines

$$\bar{\sigma}(\bar{\nu}_e p \rightarrow e^+ n) \simeq 9.5 \times 10^{-44} \mathrm{cm}^2$$

SM

- SM of particle physics is successful theory of particles and their interactions.
- It unifies strong, weak and EM interactions.
- Based on gauge group $\mathbf{SU(3)_C \times SU(2)_L \times U(1)_Y}$.
- The group $\mathbf{SU(3)_C}$ (**C**-color degree of freedom) describes strong interactions of quarks mediated through gluons.
- $\mathbf{SU(2)_L \times U(1)_Y}$ (**L** and **Y** left-handedness, hypercharge) describes the electroweak interactions.
- In SM, Higgs mechanism provides a theoretically consistent framework to generate masses for both quarks and leptons.

Masses in The Standard Model

- Fermions acquire mass after spontaneous breaking of the $\mathbf{SU(2)_L \times U(1)_Y}$ gauge symmetry to $\mathbf{U(1)_{EM}}$ by the non-zero vacuum expectation value (VEV) of Higgs field.

$$-\mathcal{L}_Y = Y_u \bar{Q}_L \tilde{\Phi} u_R + Y_d \bar{Q}_L \Phi d_R + Y_e \bar{l}_L \Phi e_R, \quad \tilde{\Phi} \equiv i\tau_2 \Phi^*$$

- Gauge symmetry breaking via the non-zero VEV of neutral Higgs field

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^o \end{pmatrix} \xrightarrow{\text{SSB}} \langle \Phi \rangle_o = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}$$

$$m_f = \frac{Y_f v}{\sqrt{2}}$$

Origin of Research Problem

- Fermion masses and mixings in the SM are generated by Yukawa coupling matrices $\mathbf{Y_u}$, $\mathbf{Y_d}$ and $\mathbf{Y_e}$ to fit the observations.
- It provides no understanding of quark and lepton masses.
- In the SM, there are no right-handed weak-isosinglet neutrinos $\nu_\mathbf{R}$ and Higgs mechanism cannot generate neutrino mass term.

Striking features of spectroscopy of quarks and leptons are:

- Similarity between the mass spectra of the three families of quarks and charged leptons.
- The charged fermion masses range over five orders of magnitude from 0.5 MeV for the electron to around 170 GeV for the top quark.

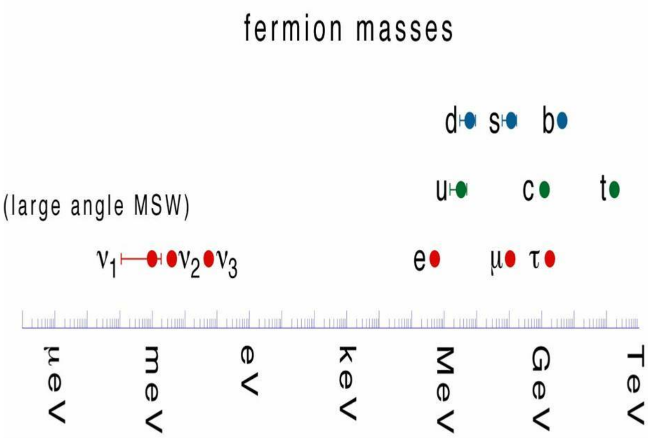


Figure: Neutrinos have tiny mass as compared to the other SM fermions.

Smallness of Neutrino Masses

- Higgs mechanism generate masses for neutrinos but suggests that neutrino masses should be of the same order as those of the other particles of the SM.
- To obtain small masses, Yukawa couplings of the neutrinos to the Higgs boson must be $\mathbf{10^{12}}$ times weaker than Yukawa couplings of the top quark.
- To explain the smallness of neutrino masses seesaw mechanisms are the prime candidates.
- They relate the smallness of neutrino masses with the existence of a very large mass scale.
- In type-I seesaw mechanism, right-handed neutrinos are added to the SM.
- Inclusion of right handed neutrinos allows for the Dirac mass term for neutrinos. Majorana mass term is possible if we abandon lepton number conservation.

Type-I seesaw

- The Dirac neutrino mass matrix ($\mathbf{M_D}$) along with the right handed Majorana mass matrix ($\mathbf{M_R}$) forms a Majorana mass term for neutrino fields:

$$\mathcal{L}_{\mathbf{M_D}+\mathbf{M_R}} = \frac{1}{2}\mathbf{N_L^T C^{-1} M N_L} + \text{ H. c.}$$

with $\mathbf{N_L} = (\nu_\mathbf{L}, (\nu_\mathbf{R})^\mathbf{c})^\mathbf{T} = (\nu_\mathbf{L}, \nu_\mathbf{L}^\mathbf{c})^\mathbf{T}$.

- The $\mathbf{6} \times \mathbf{6}$ mass matrix $\mathbf{M}$ has the form

$$\mathbf{M} = \begin{pmatrix} \mathbf{0} & \mathbf{M_D} \\ \mathbf{M_D^T} & \mathbf{M_R} \end{pmatrix}.$$

- For $\mathbf{M_R} \gg \mathbf{M_D}$, Block diagonalization of $\mathbf{M}$ yields:

$$\mathbf{M}_\nu \approx -\mathbf{M_D M_R^{-1} M_D^T}.$$

the smallness of $\mathbf{M}_\nu$ is a direct consequence of the large mass scale of $\mathbf{M_R}$.

Neutrino Mass Matrix

- Neutrino mass matrix contains all the information about neutrino masses and mixing angles.
- The neutrino mass matrix can be diagonalized as:

$$\mathrm{Diag}(m_1, m_2, m_3) = U^* P^* M_\nu P^\dagger U^\dagger$$

where $P = \mathrm{Diag}(1, e^{i\alpha}, e^{i\beta})$, α and β are the two Majorana phases and δ given below is Dirac CP-Violation phase.

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

with $s_{ij} = \sin\theta_{ij}$ and $c_{ij} = \cos\theta_{ij}$

Neutrino Mass Matrix

- The mass matrix for Majorana neutrinos contains nine physical parameters Viz. the three mass eigenvalues, three mixing angles and the three CP-Violating phases (including one Dirac phase and two Majorana phases).
- The two mass-squared differences ($\mathbf{\Delta m^2_{21}}$ and $|\mathbf{\Delta m^2_{31}}|$) and three mixing angles (solar $\boldsymbol{\theta_{12}}$, atmospheric $\boldsymbol{\theta_{23}}$ and reactor $\boldsymbol{\theta_{13}}$) have been measured in Solar, Atmospheric and Reactor experiments.
- The third mixing angle $\boldsymbol{\theta_{13}}$ was measured recently measured with a good precision[1] and its best fit is around '$\mathbf{9^o}$'.

[1] K. Abe *et al.* [T2K collaboration], *Phys. Rev. Lett.* **107**, 041801 (2011), arXiv:1106.2822 [hep-ex]; P. Adamson *et al.* [MINOS collaboration], *Phys. Rev. Lett.* **107**, 181802 (2011), arXiv:1108.0015 [hep-ex]; Y. Abe *et al.*, [Double Chooz collaboration], *Phys. Rev. Lett.* **108**, 131801 (2012), arXiv:1112.6353 [hep-ex]; F. P. An *et al.*, [Daya Bay collaboration], *Phys. Rev. Lett.* **108**, 171803 (2012), arXiv:1203.1669 [hep-ex]; Soo-Bong Kim, for RENO collaboration, *Phys. Rev. Lett.* **108**, 191802 (2012), arXiv:1204.0626 [hep-ex].

Neutrino Oscillation Data

Parameter	mean $^{(+1\sigma,+2\sigma,+3\sigma)}_{(-1\sigma,-2\sigma,-3\sigma)}$
$\Delta m^2_{21}[10^{-5}\mathrm{eV}^2]$	$7.62^{(+0.19,+0.39,+0.58)}_{(-0.19,-0.35,-0.5)}$
$\Delta m^2_{31}[10^{-3}\mathrm{eV}^2]$	$2.55^{(+0.06,+0.13,+0.19)}_{(-0.09,-0.19,-0.24)}$, $\left(-2.43^{(+0.09,+0.19,+0.24)}_{(-0.07,-0.15,-0.21)}\right)$
$\sin^2\theta_{12}$	$0.32^{(+0.016,+0.03,+0.05)}_{(-0.017,-0.03,-0.05)}$
$\sin^2\theta_{23}$	$0.613^{(+0.022,+0.047,+0.067)}_{(-0.04,-0.233,-0.25)}$, $\left(0.60^{(+0.026,+0.05,+0.07)}_{(-0.031,-0.210,-0.230)}\right)$
$\sin^2\theta_{13}$	$0.0246^{(+0.0028,+0.0056,+0.0076)}_{(-0.0029,-0.0054,-0.0084)}$, $\left(0.0250^{(+0.0026,+0.005,+0.008)}_{(-0.0027,-0.005,-0.008)}\right)$

Table: [2].

[2] M. Tortola, J. W. F. Valle and D. Vanegas, *Phys. Rev.* **D 86**, 073012 (2012), arXiv:1205.4018 [hep-ph].

Neutrino Masses

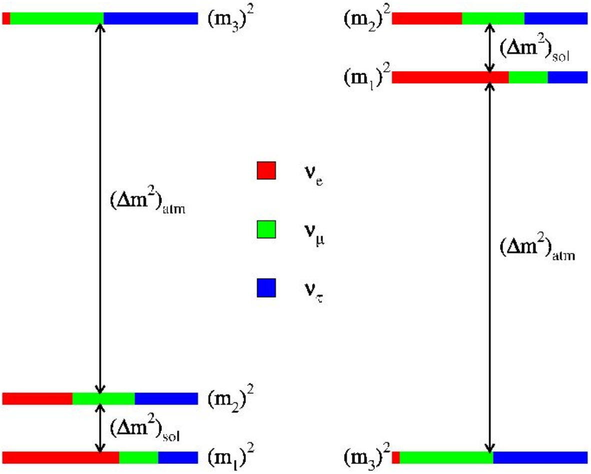


Figure: Neutrino mass and flavor spectra for the normal and inverted mass hierarchies.

Neutrino Mass Matrix

- The effective Majorana mass of the electron neutrino ($\mathbf{M_{ee}}$) which determines the rate of neutrinoless double beta (NDB) decay is given by

$$\mathbf{M_{ee} = |m_1 U_{e1}^2 + m_2 U_{e2}^2 e^{2i\alpha} + m_3 U_{e3}^2 e^{2i\beta}|.}$$

- The possible measurement of the effective Majorana mass in neutrinoless double beta decay searches will provide additional constraint on the three neutrino parameters viz. the neutrino mass scale and the two Majorana-type CP violation phases.
- But the two Majorana phases will not be uniquely determined from the effective Majorana Mass measurements even if the neutrino mass scale is known.
- Neutrino mass scale will be independently determined by the direct beta decay searches and cosmological observations.

Neutrino Mass Scale

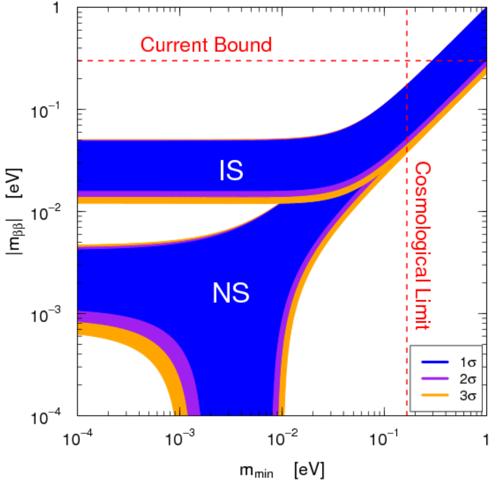


Figure: Range for $\mathbf{M_{ee}}$ as a function of smallest neutrino mass.

Neutrino Mass Matrix

- The neutrino mass matrix contains more unknown parameters than the results of the foreseeable experiments.
- Neutrino Mass Matrix is not well determined
- The strategy is to employ other theoretical inputs to reduce the number of parameters in the neutrino mass matrix. There are mainly two approaches to explain neutrino masses and mixings:
  1. Mass independent textures which lead to mixing matrices independent of the eigenvalues.
  2. Mass dependent textures which induce relations between mixing matrix elements and mass eigenvalues.
- The Most studied example of mass independent textures is Tribimaximal mixing(TBM) which predicts a vanishing 1-3 mixing angle $\theta_{13} = 0$, maximal 2-3 mixing angle $\theta_{23} = \pi/4$ and 1-2 mixing angle $\theta_{12} = \sin^{-1}(1/\sqrt{3})$.
- Some examples of mass dependent textures are zero textures, vanishing minors, hybrid textures.

Two Zero Textures in the Neutrino Mass Matrix

A_1	A_2	B_1
$\begin{pmatrix} 0 & 0 & X \\ 0 & X & X \\ X & X & X \end{pmatrix}$	$\begin{pmatrix} 0 & X & 0 \\ X & X & X \\ 0 & X & X \end{pmatrix}$	$\begin{pmatrix} X & X & 0 \\ X & 0 & X \\ 0 & X & X \end{pmatrix}$
B_2	B_3	B_4
$\begin{pmatrix} X & 0 & X \\ 0 & X & X \\ X & X & 0 \end{pmatrix}$	$\begin{pmatrix} X & 0 & X \\ 0 & 0 & X \\ X & X & X \end{pmatrix}$	$\begin{pmatrix} X & X & 0 \\ X & X & X \\ 0 & X & 0 \end{pmatrix}$
C	-	-
$\begin{pmatrix} X & X & X \\ X & 0 & X \\ X & X & 0 \end{pmatrix}$	-	-

Table: Viable two zero texture neutrino mass matrices. **X** denote the non zero elements.

Zeros from cyclic family symmetry

- Zeros in $\mathbf{M}_\nu$ having texture zeros are obtained from certain Abelian Family symmetries at expense of extended scalar sector. [3]
- We present the viability of two zeros [4] in context of Type-I and Type-II seesaw mechanism with the introduction of scalar doublet which transforms trivially under the new family symmetry $\mathbf{Z_3}/\mathbf{Z_4}$.
- $\mathbf{Z_3}$ can realize class $\mathbf{A_1}$ of Frampton zero where $\mathbf{M}_{\nu 11} = \mathbf{M}_{\nu 12}=0$ when leptonic fields transform as

$$\begin{aligned}
\mathbf{D_{L_e}} &\to \omega^2 \mathbf{D_{L_e}}, & \mathbf{e_R} &\to \omega^2 \mathbf{e_R}, & \nu_{\mathbf{R_1}} &\to \omega \nu_{\mathbf{R_1}}, \\
\mathbf{D_{L_\mu}} &\to \mathbf{D_{L_\mu}}, & \mu_{\mathbf{R}} &\to \mu_{\mathbf{R}}, & \nu_{\mathbf{R_2}} &\to \nu_{\mathbf{R_2}}, \\
\mathbf{D_{L_\tau}} &\to \omega \mathbf{D_{L_\tau}}, & \tau_{\mathbf{R}} &\to \omega \tau_{\mathbf{R}}, & \nu_{\mathbf{R_3}} &\to \nu_{\mathbf{R_3}},
\end{aligned}$$

$\omega = \exp(\mathbf{2i}\pi/\mathbf{3})$ generator of $\mathbf{Z_3}$

[3] A. S. Joshipura et al, *Eur. Phys. J.* **C 36**, 227; C. Low, *Phys. Rev.* **D 71**, 073007; W. Grimus, *Proc. Sci.*, **HEP 186** (2005)

[4] S. Dev et al *Phys. Lett B* 701: 605-608

Zeros from cyclic family symmetry

- These transformations generate diagonal $\mathbf{M_l}$
- The bilinears $\overline{\mathbf{D}}_{\mathbf{L_j}}\nu_{\mathbf{R_k}}$ and $\nu_{\mathbf{R_j}}\nu_{\mathbf{R_k}}$ relevant for $\mathbf{M_D}$ and $\mathbf{M_R}$ transform as

$$\overline{\mathbf{D}}_{\mathbf{L_j}}\nu_{\mathbf{R_k}} \sim \begin{pmatrix} \omega^2 & \omega & \omega \\ \omega & 1 & 1 \\ 1 & \omega^2 & \omega^2 \end{pmatrix}, \; \nu_{\mathbf{R_j}}\nu_{\mathbf{R_k}} \sim \begin{pmatrix} \omega^2 & \omega & \omega \\ \omega & 1 & 1 \\ \omega & 1 & 1 \end{pmatrix}.$$

$$\mathbf{M_D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & X & X \\ X & 0 & 0 \end{pmatrix}.$$

Assuming complex scalar singlet χ that transforms under $\mathbf{Z_3}$ as $\chi \to \omega^2\chi$, $\mathbf{M_R}$ becomes

$$\mathbf{M_R} = \begin{pmatrix} 0 & X & X \\ X & X & X \\ X & X & X \end{pmatrix}.$$

Zeros from cyclic family symmetry

- $$\mathbf{M}_\nu^{\mathbf{I}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & X & X \\ 0 & X & X \end{pmatrix}$$

The bilinear $\mathbf{D}_{\mathbf{L_j}}^{\mathbf{T}}\mathbf{C}^{-1}\mathbf{D}_{\mathbf{L_k}}$ relevant for $\mathbf{M_L}$ transforms as

$$\mathbf{D}_{\mathbf{L_j}}^{\mathbf{T}}\mathbf{C}^{-1}\mathbf{D}_{\mathbf{L_k}} \sim \begin{pmatrix} \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}$$

$$\mathbf{M}_\nu^{\mathbf{II}} = \begin{pmatrix} 0 & 0 & X \\ 0 & X & 0 \\ X & 0 & 0 \end{pmatrix}$$

$$\mathbf{M}_\nu = \begin{pmatrix} 0 & 0 & X \\ 0 & X & X \\ X & X & X \end{pmatrix}$$

Zeros from cyclic family symmetry

- Class $\mathbf{A_2}$ of Frampton zero $\mathbf{M_{\nu 11} = M_{\nu 13}}=0$ when leptonic fields transform as

$$\begin{array}{lll} \mathbf{D_{L_e}} \to \omega^2 \mathbf{D_{L_e}}, & \mathbf{e_R} \to \omega^2 \mathbf{e_R}, & \nu_{\mathbf{R_1}} \to \omega \nu_{\mathbf{R_1}}, \\ \mathbf{D_{L_\mu}} \to \omega \mathbf{D_{L_\mu}}, & \mu_{\mathbf{R}} \to \omega \mu_{\mathbf{R}}, & \nu_{\mathbf{R_2}} \to \nu_{\mathbf{R_2}}, \\ \mathbf{D_{L_\tau}} \to \mathbf{D_{L_\tau}}, & \tau_{\mathbf{R}} \to \tau_{\mathbf{R}}, & \nu_{\mathbf{R_3}} \to \nu_{\mathbf{R_3}} \end{array}$$

- χ transforms as $\chi \to \omega^2 \chi$ resulting in same $\mathbf{M_R}$ as $\mathbf{A_1}$ and scalar triplet is invariant under $\mathbf{Z_3}$

$$\mathbf{M_\nu^{II}} = \begin{pmatrix} \mathbf{0} & \mathbf{X} & \mathbf{0} \\ \mathbf{X} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{X} \end{pmatrix}$$

Zeros from cyclic family symmetry

For classes $\mathbf{B_1}$ $(\mathbf{M_{\nu 13} = M_{\nu 22}}=0)$, $\mathbf{B_2}$ $(\mathbf{M_{\nu 12} = M_{\nu 33}}=0)$

$$\begin{aligned} \mathbf{D_{L_e}} &\to \mathbf{D_{L_e}}, & \mathbf{e_R} &\to \mathbf{e_R}, & \nu_{\mathbf{R_1}} &\to \nu_{\mathbf{R_1}}, \\ \mathbf{D_{L_\mu}} &\to \omega \mathbf{D_{L_\mu}}, & \mu_{\mathbf{R}} &\to \omega \mu_{\mathbf{R}}, & \nu_{\mathbf{R_2}} &\to \omega \nu_{\mathbf{R_2}}, \\ \mathbf{D_{L_\tau}} &\to \omega^2 \mathbf{D_{L_\tau}}, & \tau_{\mathbf{R}} &\to \omega^2 \tau_{\mathbf{R}}, & \nu_{\mathbf{R_3}} &\to \omega^2 \nu_{\mathbf{R_3}} \end{aligned}$$

leads to diagonal $\mathbf{M_D}$

$$\mathbf{\Delta} \to \omega^2 \mathbf{\Delta} (\mathrm{forclass} \mathrm{B}_1)$$
$$\mathbf{\Delta} \to \omega \mathbf{\Delta} (\mathrm{forclass} \mathrm{B}_2)$$

$$\mathbf{M}_{\nu}^{\mathbf{II}} = \begin{pmatrix} 0 & X & 0 \\ X & 0 & 0 \\ 0 & 0 & X \end{pmatrix}$$

for class $\mathbf{B_1}$

Zeros from cyclic family symmetry

$$\mathbf{M}_{\nu}^{\mathbf{II}} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{X} \\ \mathbf{0} & \mathbf{X} & \mathbf{0} \\ \mathbf{X} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

for class $\mathbf{B_2}$. Type (I+II) seesaw yields classes $\mathbf{B_1}$ and $\mathbf{B_2}$ of the FGM two zero neutrino mass matrices

Zeros from cyclic family symmetry

For class C where $M_{\nu 22} = M_{\nu 33} = 0$

The leptonic fields are required to transform under Z_4 as

$$\begin{array}{lll} D_{L_e} \to D_{L_e}, & e_R \to e_R, & \nu_{R_1} \to \nu_{R_1}, \\ D_{L_\mu} \to iD_{L_\mu}, & \mu_R \to i\mu_R, & \nu_{R_2} \to i\nu_{R_2}, \\ D_{L_\tau} \to -iD_{L_\tau}, & \tau_R \to -i\tau_R, & \nu_{R_3} \to -i\nu_{R_3} \end{array}$$

leading to diagonal M_l, M_D and non diagonal M_R.

$$M_\nu^l = \begin{pmatrix} X & 0 & 0 \\ 0 & 0 & X \\ 0 & X & 0 \end{pmatrix}$$

$$\Delta_1 \to i\Delta_1$$
$$\Delta_2 \to -i\Delta_2$$

Zeros from cyclic family symmetry

$$\mathbf{M}_\nu^{\mathbf{II}} = \begin{pmatrix} \mathbf{0} & \mathbf{X} & \mathbf{X} \\ \mathbf{X} & \mathbf{0} & \mathbf{0} \\ \mathbf{X} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

We introduced three right-handed neutrino singlets and at the most two scalar triplets which acquire small nonzero VEVs.

However, in some cases we, introduced an additional complex scalar singlet which transforms nontrivially under $\mathbf{Z_3}$ symmetry.

$\mathbf{M}_\nu$ with two vanishing minors

- Two vanishing minors of $\mathbf{M}_\nu$ (15 possibilities) in the flavor basis and large $\mathbf{M_{ee}}$ [5]
- Only 7 can accomodate neutrino oscillation data,

  Three cases $\mathbf{B_5}$ ($\mathbf{C_{3,3}}$, $\mathbf{C_{1,2}}$), $\mathbf{B_6}$ ($\mathbf{C_{2,2}}$, $\mathbf{C_{1,3}}$) and $\mathbf{D}$ ($\mathbf{C_{3,3}}$, $\mathbf{C_{2,2}}$) provide non-trivial zero minors, other reduce to 2 zero when confronted with the neutrino oscillation data
- We work in basis where $\mathbf{M_D}$ is diagonal ($\mathbf{M_D = diag(x, y, z)}$), and the neutrino mixing arises solely from $\mathbf{M_R}$. A zero entry in $\mathbf{M_R}$ propagates as vanishing minor in $\mathbf{M}_\nu$
- To obtain $\mathbf{B_5}$, $\mathbf{B_6}$, we extend SM by adding three ν_{R_i} and one scalar singlet χ

$$\mathbf{M_R(B_5)} = \begin{pmatrix} a & 0 & c \\ 0 & d & e \\ c & e & 0 \end{pmatrix}, \quad \mathbf{M_R(B_6)} = \begin{pmatrix} a & b & 0 \\ b & 0 & e \\ 0 & e & f \end{pmatrix}$$

[5] S. Dev et al *PLB* 76, 168-176

M_ν with two vanishing minors

$$M_\nu(B_5) = \frac{1}{c^2d + ae^2}\begin{pmatrix} e^2x^2 & -cexy & cdxz \\ -cexy & c^2y^2 & aeyz \\ cdxz & aeyz & -adz^2 \end{pmatrix}, B_6 = \frac{1}{b^2f + ae^2}\begin{pmatrix} e^2x^2 & bfxy & -bexz \\ bfxy & -afy^2 & aeyz \\ -bexz & aeyz & b^2z^2 \end{pmatrix}$$

- Symmetry realization of $\mathbf{B_5}$ and $\mathbf{B_6}$ we consider cyclic group $\mathbf{Z_3}$ which is minimal group as $\mathbf{Z_2}$ leads to non-diagonal $\mathbf{M_l}$ and $\mathbf{M_D}$.

$$\begin{aligned} D_{L_1} &\to D_{L_1}, & l_{R1} &\to l_{R1}, & \nu_{R_1} &\to \nu_{R_1}, \\ D_{L_2} &\to \omega D_{L_2}, & l_{R2} &\to \omega l_{R2}, & \nu_{R_2} &\to \omega\nu_{R_2}, \\ D_{L_3} &\to \omega^2 D_{L_3}, & l_{R3} &\to \omega^2 l_{R3}, & \nu_{R_3} &\to \omega^2\nu_{R_3}, \end{aligned}$$

- Bilinears $\overline{D}_{L_j} l_{R_k}$, $\overline{D}_{L_j}\nu_{R_k}$, relevant for $\mathbf{M_l}$ and $\mathbf{M_D}$ transform as

$$\overline{D}_{L_j} l_{R_k} \sim \overline{D}_{L_j}\nu_{R_k} \sim \begin{pmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{pmatrix}$$

$\mathbf{M}_\nu$ with two vanishing minors

- SM Higgs doublet remains invariant under $\mathbf{Z_3}$ leading to diagonal $\mathbf{M_l}$ and $\mathbf{M_D}$ bilinear $\nu_{R_j}\nu_{R_k}$ relevant for $\mathbf{M_R}$ transforms as

$$\nu_{R_j}\nu_{R_k} \sim \begin{pmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{pmatrix}$$

- $\chi \to \omega\chi$, $\mathbf{B_5}$ and $\chi \to \omega^2\chi$ for class $\mathbf{B_6}$ which leads to $\mathbf{Z_3}$ invariant Yukawa Lagrangians

$$-\mathcal{L}_{(B_5)} = Y^l_{11}\overline{D}_{L_1}\phi l_{R_1} + Y^l_{22}\overline{D}_{L_2}\phi l_{R_2} + Y^l_{33}\overline{D}_{L_3}\phi l_{R_3} + Y^D_{11}\overline{D}_{L_1}\tilde{\phi}\nu_{R_1}$$
$$+Y^D_{22}\overline{D}_{L_2}\tilde{\phi}\nu_{R_2} + Y^D_{33}\overline{D}_{L_3}\tilde{\phi}\nu_{R_3} + \frac{Y^M_{13}}{2}\nu^T_{R_1}C^{-1}\nu_{R_3}\chi$$
$$+\frac{Y^M_{22}}{2}\nu^T_{R_2}C^{-1}\nu_{R_2}\chi + \frac{M^M_{11}}{2}\nu^T_{R_1}C^{-1}\nu_{R_1} + \frac{M^M_{23}}{2}\nu^T_{R_2}C^{-1}\nu_{R_3} + \text{H.c.}$$

$$-\mathcal{L}_{(B_6)} = Y^l_{11}\overline{D}_{L_1}\phi l_{R_1} + Y^l_{22}\overline{D}_{L_2}\phi l_{R_2} + Y^l_{33}\overline{D}_{L_3}\phi l_{R_3} + Y^D_{11}\overline{D}_{L_1}\tilde{\phi}\nu_{R_1}$$
$$+Y^D_{22}\overline{D}_{L_2}\tilde{\phi}\nu_{R_2} + Y^D_{33}\overline{D}_{L_3}\tilde{\phi}\nu_{R_3} + \frac{Y^M_{12}}{2}\nu^T_{R_1}C^{-1}\nu_{R_2}\chi + \frac{Y^M_{33}}{2}\nu^T_{R_3}C^{-1}\nu_{R_3}\chi$$
$$+\frac{M^M_{11}}{2}\nu^T_{R_1}C^{-1}\nu_{R_1} + \frac{M^M_{23}}{2}\nu^T_{R_2}C^{-1}\nu_{R_3} + \text{H.c.}$$

$\mathbf{M}_\nu$ with two vanishing minors

- $\mathbf{M_R}$ has 2 mass terms
  Bare mass term (do not need a scalar singlet and is invariant by itself)

  Terms arising from Yukawa couplings to χ It's restricted by the scale of $\mathbf{Z_3}$ breaking while no such restriction on bare mass term which can have a higher mass scale
- **ee** and $\mu\tau$ entries of $\mathbf{M}_\nu$ have contributions to their numerators from **ee** and $\mu\tau$ entries of $\mathbf{M_R}$ which arise from the bare mass term

  We assume mass EV of $\mathbf{M_D}$ to be of same order of magnitude which leads to a large value of **ee** and $\mu\tau$ entries of $\mathbf{M}_\nu$ while the other elements of $\mathbf{M}_\nu$ are suppressed, thus, leading to a large value of $\mathbf{M_{ee}}$

Formalism

- The simultaneous existence of two vanishing minors in the neutrino mass matrix implies

$$M_{\nu(pq)}M_{\nu(rs)} - M_{\nu(tu)}M_{\nu(vw)} = 0$$
$$M_{\nu(p'q')}M_{\nu(r's')} - M_{\nu(t'u')}M_{\nu(v'w')} = 0$$

The mass ratios for class $\mathbf{B_5}$, $\mathbf{B_6}$ to 1st order $\mathbf{s_{13}}$

$$\frac{m_1}{m_2}e^{-2i\alpha} \approx 1 + \frac{s_{13}s_{23}\left(c_{23}^2 e^{-i\delta} + s_{23}^2 e^{i\delta}\right)}{c_{12}c_{23}^3 s_{12}}$$

$$\frac{m_1}{m_3}e^{-2i\beta} \approx -\frac{s_{23}^2 e^{2i\delta}}{c_{23}^2} - \frac{c_{12}s_{13}s_{23}^3\left(c_{23}^2 e^{-i\delta} + s_{23}^2 e^{i\delta}\right)e^{2i\delta}}{c_{23}^5 s_{12}} \, .$$

$$\frac{m_1}{m_2}e^{-2i\alpha} \approx 1 - \frac{s_{13}c_{23}\left(c_{23}^2 e^{i\delta} + s_{23}^2 e^{-i\delta}\right)}{c_{12}s_{23}^3 s_{12}}$$

$$\frac{m_1}{m_3}e^{-2i\beta} \approx -\frac{c_{23}^2 e^{2i\delta}}{s_{23}^2} + \frac{c_{12}s_{13}c_{23}^3\left(c_{23}^2 e^{i\delta} + s_{23}^2 e^{-i\delta}\right)e^{2i\delta}}{s_{23}^5 s_{12}} \, .$$

Formalism

$$M_R^{B_6} = P_{23} M_R^{B_5} P_{23}^T$$

which gives

$$M_\nu^{B_6} = P_{23} M_\nu^{B_5} P_{23}^T$$

$$\theta_{12}^{B_6} = \theta_{12}^{B_5},\ \theta_{13}^{B_6} = \theta_{13}^{B_5},\ \theta_{23}^{B_6} = \frac{\pi}{2} - \theta_{23}^{B_5},\ \delta^{B_6} = \delta^{B_5} - \pi.$$

$$\rho = \left| \frac{m_1}{m_3} e^{-2i\beta} \right|,$$

$$\sigma = \left| \frac{m_1}{m_2} e^{-2i\alpha} \right|.$$

while the CP- violating Majorana phases α and β are given by

$$\alpha = -\frac{1}{2} \arg \left(\frac{A_3 B_1 - A_1 B_3}{A_2 B_3 - A_3 B_2} \right),$$

$$\beta = -\frac{1}{2} \arg \left(\frac{A_2 B_1 - A_1 B_2}{A_3 B_2 - A_2 B_3} e^{2i\delta} \right).$$

Formalism

$$A_h = (U_{pl}U_{ql}U_{rk}U_{sk} - U_{tl}U_{ul}U_{vk}U_{wk}) + (l \leftrightarrow k)$$

$$B_h = (U_{p'l}U_{q'l}U_{r'k}U_{s'k} - U_{t'l}U_{u'l}U_{v'k}U_{w'k}) + (l \leftrightarrow k)$$

$$m_1 = \sigma\sqrt{\frac{\Delta m^2_{12}}{1-\sigma^2}}\,, \quad m_1 = \rho\sqrt{\frac{\Delta m^2_{12} + |\Delta m^2_{23}|}{1-\rho^2}}\,.$$

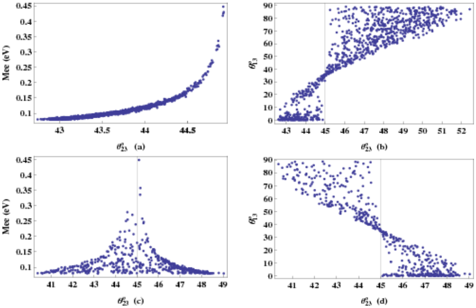


Figure: Correlation plots for class $\mathbf{B_5}$, for Normal Spectrum (NS) and Inverted Spectrum (IS).

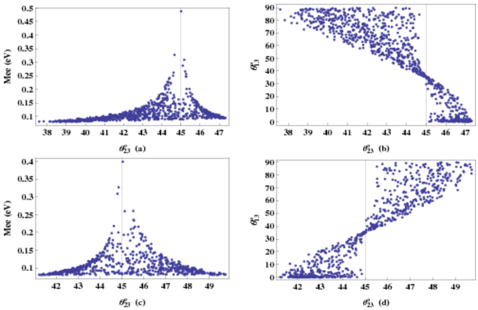
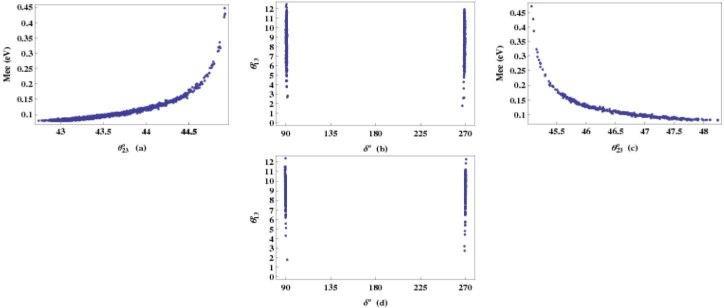


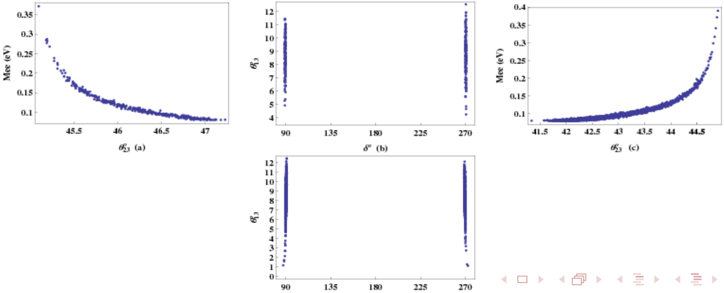
Figure: Correlation plots for class $\mathbf{B_6}$,for NS and IS.

- Classes $\mathbf{B_5}$ and $\mathbf{B_6}$ predict a near maximal atmospheric mixing angle while the other two mixing angles remain unconstrained.
- θ_{23} moves towards $\pi/4$ with increasing $\mathbf{M_{ee}}$ for $\mathbf{B_5}$, $\mathbf{B_6}$. $\mathbf{B_5}$ and $\mathbf{B_6}$ of two vanishing minors in $\mathbf{M_\nu}$ naturally predict a near maximal atmospheric mixing angle.

Formalism



Mee (eV)
θ°23 (a)
θ°13
δ° (b)
Mee (eV)
θ°23 (c)
θ°13
δ° (d)



Mee (eV)
θ°23 (a)
θ°13
δ° (b)
Mee (eV)
θ°23 (c)
θ°13

Results

$$M_\nu^{B_5}(NS) = \begin{pmatrix} 0.112956 - 0.000905i & 0.000205 - 0.002058i & 0.000018 - 0.000166i \\ 0.000205 - 0.002058i & -0.000037 - 0.000008i & -0.117485 + 0.002109i \\ 0.000018 - 0.000166i & -0.117485 + 0.002109i & -0.009482 + 0.000114i \end{pmatrix},$$

$$M_\nu^{B_5}(IS) = \begin{pmatrix} 0.127210 - 0.001176i & 0.000147 - 0.002268i & -0.000012 + 0.000177i \\ 0.000147 - 0.002268i & -0.000040 - 0.000006i & -0.122682 + 0.002478i \\ -0.000012 + 0.000177i & -0.122682 + 0.002478i & 0.009575 - 0.000154i \end{pmatrix}.$$

- pattern $\mathbf{B_6}$ can be obtained from above matrices with the operation of 2-3 permutation symmetry
- taking $\mathbf{M_l}$ and $\mathbf{M_D}$ to be diagonal, the zeros of $\mathbf{M_R}$ propagate as zero minors of $\mathbf{M_\nu}$ and the origin of neutrino mixing is solely from $\mathbf{M_R}$ which can be realized be $\mathbf{Z_3}$ symmetry
- θ_{23} approaches $\pi/4$ with the increasing value of $\mathbf{M_{ee}}$. θ_{13} equal to zero is not allowed in these textures, thus, naturally accommodating a non-zero θ_{13} as suggested by the recent results of the T2K experiment

Flavor Symmetries

- Experiments T2K, and Double CHOOZ [6], Daya Bay and RENO [7] predicts θ_{13}, is not only non-zero but "large".

- Different mixing schemes which arise from form diagonalizable form

  BM $\theta_{12} = \pi/4$
  TBM $\theta_{12} = \mathrm{Sin}^{-1}\frac{1}{\sqrt{3}}$
  HM $\theta_{12} = \pi/6$
  GR1 $\theta_{12} = \tan^{-1}(1/\varphi)$
  GR2 $\theta_{12} = \cos^{-1}(\varphi/2)$,
  $\varphi = (1+\sqrt{5})/2$

$\theta_{23} = \pi/4$, $\theta_{13} = 0$

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[6] H. De. Kerrect [Double CHOOZ Collaboration], talk at LowNu conference in Korea (2011)

[7] F. P. An *et al.*, Phys. Rev. Lett. **108**, 171803 (2012); J. K. Ahn *et al.* [RENO Collaboration], Phys. Rev. Lett. **108**, 191802 (2012)

$$U = \begin{pmatrix} c'_{12} & s'_{12} & 0 \\ \frac{-s'_{12}}{\sqrt{2}} & \frac{c'_{12}}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-s'_{12}}{\sqrt{2}} & \frac{c'_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \wp$$

To generate non zero θ_{13} and non maximal θ_{23} we consider deviations from charged lepton sector [8]

$$U = e^{i\Phi} P \tilde{U} Q$$

$$P = \mathrm{diag}(1, e^{i\phi_1}, e^{i\phi_2})$$
$$Q = \mathrm{diag}(1, e^{i\rho_1}, e^{i\rho_2})$$

$\tilde{U}$ has three angles and one phase

- For mass independent textures of M_ν and $M_\nu M_\nu^\dagger$ are diagonalized by same diagonalizing matrix
- the deviation of M_l from diagonal form is considered arbitrary

[8] S Dev et al *PLB* 704 527-533

Formalism

- By taking hermitian products $\mathbf{M_l M_l^\dagger}$ and $\mathbf{M_\nu M_\nu^\dagger}$ we remove Majorana phases and one phase from $\mathbf{U}$

$$\mathbf{M_l = U_l M_l^d U_R^\dagger}$$

$$\mathbf{M_\nu = U_\nu M_\nu^d U_\nu^T}$$

$$\begin{aligned}\mathbf{M_l M_l^\dagger} &= \mathbf{U_l M_l^d U_R^\dagger U_R M_l^d U_l^\dagger} \\ &= \mathbf{U_l (M_l^d)^2 U_l^\dagger}\end{aligned}$$

$$\begin{aligned}\mathbf{M_l M_l^\dagger} &= \mathbf{e^{i\phi_l} P_l \tilde{U}_l Q_l (M_l^d)^2 Q_l^\dagger \tilde{U}_l{}^\dagger P_l^\dagger e^{-i\phi_l}} \\ &= \mathbf{P_l \tilde{U}_l (M_l^d)^2 \tilde{U}_l{}^\dagger P_l^\dagger}\end{aligned}$$

Formalism

$$\mathbf{M}_\nu \mathbf{M}_\nu^\dagger = \tilde{\mathbf{P}}_\nu \tilde{\mathbf{U}}_\nu (\mathbf{M}_\nu^d)^2 \tilde{\mathbf{U}}_\nu^\dagger \tilde{\mathbf{P}}_\nu^\dagger$$

Absorb 2 phases from $\mathbf{P}_l$ and 1 from $\tilde{\mathbf{P}}_\nu$ in left handed lepton fields

$$\mathbf{U} = \tilde{\mathbf{U}}_l^\dagger \tilde{\mathbf{P}}_\nu \tilde{\mathbf{U}}_\nu$$

$\tilde{\mathbf{U}}_l$, $\tilde{\mathbf{U}}_\nu$ 3 real parameters, and one phase each

$\tilde{\mathbf{P}}_\nu = \mathrm{diag}(\mathbf{1}, \mathbf{1}, \mathbf{e}^{i\phi})$

Thus, U has 6 real parameters and 3 phases

Charged Lepton Corrections

$$\tilde{U}_\nu = \begin{pmatrix} c'_{12} & s'_{12} & 0 \\ \frac{-s'_{12}}{\sqrt{2}} & \frac{c'_{12}}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-s'_{12}}{\sqrt{2}} & \frac{c'_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\tilde{U}_l = \begin{pmatrix} 1 & \epsilon_{12} & e^{-i\delta_{13}}\epsilon_{13} \\ -\epsilon_{12} & 1 & \epsilon_{23} \\ -e^{i\delta_{13}}\epsilon_{13} & -\epsilon_{23} & 1 \end{pmatrix}$$

$\sin\epsilon_{ij} \approx \epsilon_{ij}, \ \cos\epsilon_{ij} \approx 1 \quad \epsilon_{ij} < 0.227$

$$U = \tilde{U}_l^\dagger P_\nu \tilde{U}_\nu$$

Charged Lepton Corrections

$$\begin{pmatrix} c'_{12} & s'_{12} & 0 \\ \frac{-s'_{12}}{\sqrt{2}} & \frac{c'_{12}}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-e^{i\phi}s'_{12}}{\sqrt{2}} & \frac{e^{i\phi}c'_{12}}{\sqrt{2}} & \frac{e^{i\phi}}{\sqrt{2}} \end{pmatrix} +$$

$$\begin{pmatrix} \frac{s'_{12}(\epsilon_{12}+e^{-i(\delta_{13}-\phi)}\epsilon_{13})}{\sqrt{2}} & -\frac{c'_{12}(\epsilon_{12}+e^{-i(\delta_{13}-\phi)}\epsilon_{13})}{\sqrt{2}} & \frac{\epsilon_{12}-e^{-i(\delta_{13}-\phi)}\epsilon_{13}}{\sqrt{2}} \\ c'_{12}\epsilon_{12} + \frac{e^{i\phi}s'_{12}\epsilon_{23}}{\sqrt{2}} & s'_{12}\epsilon_{12} - \frac{e^{i\phi}c'_{12}\epsilon_{23}}{\sqrt{2}} & \frac{-e^{i\phi}\epsilon_{23}}{\sqrt{2}} \\ \frac{-s'_{12}\epsilon_{23}}{\sqrt{2}} + e^{i\delta_{13}}c'_{12}\epsilon_{13} & \frac{c'_{12}\epsilon_{23}}{\sqrt{2}} + e^{i\delta_{13}}s'_{12}\epsilon_{13} & \frac{-\epsilon_{23}}{\sqrt{2}} \end{pmatrix}$$

Mixing angles are

$$\sin^2\theta_{13} = |U_{e3}|^2, \quad \sin^2\theta_{23} = \frac{|U_{\mu 3}|^2}{|U_{\mu 3}|^2 + |U_{\tau 3}|^2},$$

$$\sin^2\theta_{12} = \frac{|U_{e2}|^2}{|U_{e1}|^2 + |U_{e2}|^2}.$$

Charged Lepton Corrections

$$\sin\theta_{13} = \frac{\epsilon_{13} - \epsilon_{12}\cos(\delta_{13} - \phi)}{\sqrt{2}},$$

$$\sin\theta_{23} = \frac{1 + \epsilon_{23}\cos\phi}{\sqrt{2}},$$

$$\sin\theta_{12} = s'_{12} - \frac{c'_{12}(\epsilon_{12} + \epsilon_{13}\cos(\delta_{13} - \phi))}{\sqrt{2}}.$$

θ_{13} and θ_{23} are independent of θ'_{12}

Allowed ranges of perturbation parameters

$$-0.22 < (\epsilon_{13}, \epsilon_{23}) < 0.22$$

$$-0.20 < \epsilon_{12} < 0.17 \quad \text{TBM}$$

$$-0.165 < \epsilon_{12} < 0.22 \quad \text{GR1}$$

$$-0.22 < \epsilon_{12} < 0.17 \quad \text{GR2}$$

$$-0.15 < \epsilon_{12} < 0.22 \quad \text{HM}$$

$$-0.22 < \epsilon_{12} < 0 \quad \text{BM}.$$

Charged Lepton Corrections

The Jarskog CP violation rephasing invariant

$$J_{CP} = Im(U_{e2}U^*_{e3}U^*_{\mu 2}U_{\mu 3}) = Im(U_{\tau 2}U^*_{\tau 3}U^*_{e2}U_{e3})$$

$$J_{CP} = \frac{\sin 2\theta'_{12}\epsilon_{13}\sin(\delta_{13}-\phi)}{4\sqrt{2}}.$$

Single phase difference which is relevant Dirac type CP phase

Probability of oscillation between different flavors

$$P(\nu_\mu \to \nu_e) = \frac{\triangle^2_{21}\sin^2 2\theta'_{12}}{2} + (\epsilon^2_{12}+\epsilon^2_{13})\sin^2\triangle_{31}$$
$$-2\epsilon_{12}\epsilon_{13}\cos(\delta_{13}-\phi)\sin^2\triangle_{31} - \frac{\triangle_{21}\sin 2\theta'_{12}\epsilon_{13}\sin(\delta_{13}-\phi)}{\sqrt{2}}$$

Deviation from θ_{23} from maximality do not appear upto second order.

θ_{13} can have large deviation from zero irrespective of deviation of θ_{23} from maximality

Charged Lepton Corrections

- Deviations from maximal atmospheric mixing and vanishing reactor mixing are obtained through charged lepton corrections in terms of small perturbation parameters.
- Relatively large deviations for the reactor mixing angle from zero as indicated by T2K experiment can be obtained in this parametrization
- Reduce the number of complex phases to two by considering the Hermitian products of charged lepton and neutrino mass matrices
- In this parametrization θ_{13} can have a large deviation from zero irrespective of the deviation of θ_{23} from maximality to a good approximation

$\mu - \tau$ Symmetry

- Irrespective of any underlying model, an effective $\mathbf{Z_2}$ symmetry is necessary and sufficient for obtaining $\theta_{13} = 0$ [9]

$$\mathbf{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta_{23} & \sin 2\theta_{23} \\ 0 & \sin 2\theta_{23} & -\cos 2\theta_{23} \end{pmatrix},$$

- Invariance of $\mathbf{M_{\nu f}}$ under $\mathbf{S}$ leads to vanishing θ_{13}
- Motivated special case of $\mathbf{S}$ is celebrated $\mu - \tau$ symmetry

$$\mathbf{S_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

[9] S. Gupta et al hep-ph 1301.7130

$\mu - \tau$ Symmetry

- Two different scenarios
  μ-τ is effective symmetry of $\mathbf{M_{\nu f}}$. May be accidental or consequence of some broken symmetry. Here diagonal $\mathbf{M_l}$ breaks the symmetry.

  A more fundamental approach where it is approximate symmetry of $\mathbf{M_l}$ and $\mathbf{M_\nu}$
- We discuss viability or otherwise of both scenario purely from phenomenological consideration
  Approximately $\mu - \tau$ symmetry
- We define $\mu - \tau$ as requiring eigenvector of $\mathbf{M_{\nu f}}$ corresponding to heaviest (lightest) mass EV

$$\begin{pmatrix} 0 \\ \pm\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\mathcal{M}_{\nu f} = \begin{pmatrix} X & A & \mp A \\ A & B & C \\ \mp A & C & \mp B \end{pmatrix}$$

- All parameters are complex. X and C can be made real by redefinition of charged lepton mass eigenstates
- 6 parameters and 2 predictions among 8 relevant observables.

$$\mathcal{M}_{\nu f} = \begin{pmatrix} X & A & A^* \\ A & B & C \\ A^* & C & B^* \end{pmatrix}$$

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$$\theta_{23} = \frac{\pi}{4} \quad \text{and} \quad \mathrm{Re}(\cos\theta_{12}\sin\theta_{12}\sin\theta_{13}e^{i\delta}) = 0 \ .$$

Phenomenologically allowed and still a exact symmetry

[10] W. Grimus and L. Lavoura, Phys. Lett. B **572**, 189 (2003); Phys. Lett. B **579**, 113 (2004)

Approximately $\mu - \tau$ Symmetric $\mathbf{M}_{\nu f}$

- We study effects of perturbations on the former μ-τ symmetry
- characterize breaking in terms of two complex parameters

$$\epsilon_1 \equiv \frac{(\mathcal{M}_{\nu f})_{12} - (\mathcal{M}_{\nu f})_{13}}{(\mathcal{M}_{\nu f})_{12} + (\mathcal{M}_{\nu f})_{13}} \; ; \quad \epsilon_2 \equiv \frac{(\mathcal{M}_{\nu f})_{22} - (\mathcal{M}_{\nu f})_{33}}{(\mathcal{M}_{\nu f})_{22} + (\mathcal{M}_{\nu f})_{33}} \; .$$

Approximate μ-τ symmetry $|\epsilon_1|$**and**$|\epsilon_2| \ll \mathbf{1}$
Most general breaking of μ-τ symmetry and all other element of arbitrary perturbation matrix are absorbed in $\mathbf{M}_{\nu f}$

$$\mathcal{M}_{\nu f} = \mathbf{U}^* \mathrm{Diag.}(\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3) \mathbf{U}^\dagger \; ,$$

$$\epsilon_1 = \frac{\mathbf{y} + \mathbf{s}_{13}\mathbf{f}}{\mathbf{1} - \mathbf{s}_{13}\mathbf{y}\mathbf{f}} \; ,$$

$$\epsilon_2 = \frac{(\mathbf{c}_{23}^2 - \mathbf{s}_{23}^2)\mathbf{g}_- + 4\mathbf{c}_{12}\mathbf{s}_{12}\mathbf{c}_{23}\mathbf{s}_{23}\mathbf{s}_{13}\mathbf{e}^{-i\delta}(-\mathbf{m}_1 + \mathbf{m}_2\mathbf{e}^{-i\alpha_2})}{\mathbf{g}_+} \; ,$$

Approximately $\mu - \tau$ Symmetric $\mathbf{M}_{\nu f}$

$$\mathbf{f} \equiv \frac{\mathbf{m_3 e^{-i(\alpha_3-\delta)} - m_1 c_{12}^2 e^{-i\delta} - m_2 s_{12}^2 e^{-i(\alpha_2+\delta)}}}{\mathbf{s_{12} c_{12} (m_1 - m_2 e^{-i\alpha_2})}} \,,$$

$$\mathbf{g_\pm \equiv \pm m_3 e^{-i\alpha_3} c_{13}^2 + m_1 (s_{12}^2 \pm c_{12}^2 s_{13}^2 e^{-2i\delta}) + m_2 e^{-i\alpha_2} (c_{12}^2 \pm s_{12}^2 s_{13}^2 e^{-2i\delta})} \,.$$

- Here $\mathbf{s_{13}}$ and $\mathbf{y \equiv (c_{23} - s_{23})/(c_{23} + s_{23})}$ are μ-τ breaking observables. $\mathbf{-0.18 \leq y \leq 0.16}$, $\mathbf{0.12 \leq s_{13} \leq 0.17}$ (small) But $\mathbf{f}$ and $\mathbf{g_\pm}$ can be large (depend on NMH and CP phases)
- ϵ_1 plays a major role to allow or disallow μ-τ symmetry. Neglecting second term in denominator

$$\epsilon_1 \approx \mathbf{y + s_{13} f} \,.$$

. ϵ_1 is small if $|\mathbf{f}| \sim \mathcal{O}(1)$.

Approximately $\mu - \tau$ Symmetric $\mathbf{M}_{\nu f}$

(A) Normal hierarchy: $\mathbf{m_1 \ll m_2 \approx \sqrt{\Delta_\odot} \ll m_3 \approx \sqrt{\Delta_A}}$

$$\mathbf{f \approx -\frac{\sqrt{\Delta_A/\Delta_\odot}}{s_{12}c_{12}} e^{i(\alpha_2-\alpha_3+\delta)} \left(1 + \mathcal{O}\left(\frac{\Delta_\odot}{\Delta_A}\right)\right)}\,,\ \mathbf{|f| \approx 12.5(1+\mathcal{O}(0.2))},$$

leads to $|\epsilon_1| \approx \mathbf{|y + s_{13}f| \geq 1.6}$

(B) Inverted hierarchy:
$\mathbf{m_1 \approx \sqrt{\Delta_A}, m_2 \approx \sqrt{\Delta_\odot + \Delta_A} \gg m_3}$

$$\mathbf{f \approx -e^{-i\delta} \frac{c_{12}^2 + s_{12}^2 e^{-i\alpha_2} + \mathcal{O}(\Delta_\odot/\Delta_A)}{s_{12}c_{12}(1 - e^{-i\alpha_2} + \mathcal{O}(\Delta_\odot/\Delta_A))}}\,.$$

$\mathbf{f}$ is enhanced for $\alpha_2 \sim \mathbf{0}$ leading to large ϵ_1,
$\mathcal{O}(\mathbf{\cot 2\theta_{12}})$ for $\alpha_2 \sim \pi$. Allowed range of α_2 is close to $\pi/2 < \alpha_2 < \pi$ for which $|\epsilon_1| \leq \mathbf{0.2}$.

(C) Quasi degeneracy:

$\mathbf{m_1 = m_0 \gg \sqrt{\Delta_\odot}, m_2 = \sqrt{m_0^2 + \Delta_\odot}, m_3 = \sqrt{m_0^2 + \Delta_A}}$

allowed $|\mathbf{f}|$ can be obtained by considering limiting cases of the Majorana phases corresponding to CP conserving situation. 4 possibilities with initial signs of the three masses: (i) + + +, (ii) + + -, (iii) + - + and (iv) + - -

$$\mathbf{f} \approx \frac{\pm(1 + \frac{\Delta_A}{2m_0^2})e^{i\delta} - e^{-i\delta}}{\frac{\Delta_\odot}{2m_0^2} c_{12} s_{12}} \qquad \text{(i), (ii)} ,$$

$$\mathbf{f} \approx \frac{1}{\sin 2\theta_{12}} \left(\pm \left(1 + \frac{\Delta_A}{2m_0^2} \right) e^{i\delta} - \cos 2\theta_{12} e^{-i\delta} \right) \qquad \text{(iii), (iv)}$$

$|\mathbf{f}|$ large $\geq \mathbf{\Delta_A / \Delta_\odot}$ for (i), (ii) while for (iii) and (iv), $|\mathbf{f}| < \cot\theta_{12}$ and maximum value is attained for $\delta = \pi/2(0)$ for (iii) (iv). Both cases allow small ϵ_1.

To find range of viability of μ-τ symmetry, we also needs to consider ϵ_2 and allow non-trivial phases.

Approximately $\mu - \tau$ Symmetric $\mathbf{M}_{\nu f}$

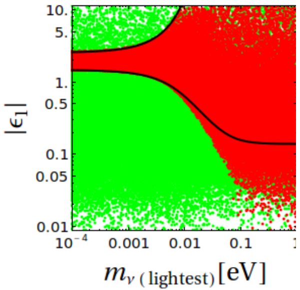


Figure: Allowed $|\epsilon_1|$, $|\epsilon_2|$ for NH(red), IH(green) in neutrino masses varying δ and $\alpha_{2,3}$ for central values of other observables.The CP conserving cases for the normal hierarchy: $\delta = \alpha_2 = 0$, $\alpha_3 = \pi$ (upper line) and $\delta = 0$, $\alpha_2 = \alpha_3 = \pi$ (lower line).

$|\epsilon_1| > |\epsilon_2|$ at given $\mathbf{m_0}$ for NH and μ-τ breaking can be small ($|\epsilon_1| \leq \mathbf{0.2}$) only for $\mathbf{m_0} > \mathbf{0.025}$ eV. Constraint $|\epsilon_1|$, $|\epsilon_2| \leq \mathbf{0.2}$ disfavors $\mathbf{m_0} < \mathbf{0.01}$ eV

Approximately $\mu - \tau$ Symmetric $\mathbf{M}_{\nu f}$

Slight increase in $|\epsilon_{1,2}|$ allows IH as $\mathbf{m_3} \sim \mathbf{0}$ eV. Thus, QD and IH give viable alternative for μ-τ symmetry to remain an approximate symmetry of $\mathcal{M}_{\nu f}$

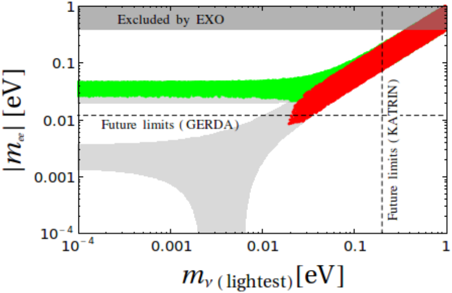


Figure: The scattered points corresponds to $|\epsilon_1|,\ |\epsilon_2| \leq \mathbf{0.3}$ while the shaded region corresponds to $|\epsilon_{1,2}|$without restriction.

$$|\mathbf{m_{ee}}| \equiv |\sum \mathbf{U^2_{ei} m_i}| \geq \mathbf{0.01}\ \text{eV}$$

Approximately $\mu - \tau$ Symmetric $\mathbf{M_l}$ and $\mathbf{M_\nu}$

- μ-τ symmetry in $\mathbf{M_{\nu f}}$ need not imply that it's present at fundamental level. $\mathbf{A_4}$ group imposed as symmetry of Lagrangian which does not have μ-τ sym. Spontaneous breaking in specific manner leads to μ-τ sym in $\mathbf{M_\nu f}$
- μ-τ sym at the fundamental level. We assume $\mathbf{M_l}$ and $\mathbf{M_\nu}$ are μ-τ sym in suitable basis

$$S_2^T M_\nu S_2 = M_\nu \ ,$$
$$S_2^T M_l M_l^\dagger S_2 = M_l M_l^\dagger \ ,$$
$$U_l^T M_l M_l^\dagger U_l = D_l$$
$$U_l = R_{23}(\pi/4) U_{12} \ .$$

$\mathcal{M}_{\nu f} \equiv U_l^T M_\nu U_l$ satisfies

$$\tilde{S}_2^T \mathcal{M}_{\nu f} \tilde{S}_2 = \mathcal{M}_{\nu f} \ ,$$
$$\tilde{S}_2 \equiv U_l^\dagger S_2 U_l = \mathrm{Diag.}(1, 1, -1).$$

Approximately $\mu - \tau$ Symmetric $\mathbf{M_l}$ and $\mathbf{M_\nu}$

- Imposition of μ-τ on $\mathbf{M_l}$ and $\mathbf{M_\nu}$ is equivalent to imposing $\tilde{\mathbf{S}}_2$ on $\mathcal{M}_{\nu f}$
- Most general $\mathbf{M}_{\nu f}$ invariant under $\tilde{\mathbf{S}}_2$

$$\mathcal{M}_{\nu f} = \begin{pmatrix} x & a & 0 \\ a & b & 0 \\ 0 & 0 & c \end{pmatrix} .$$

Not viable since only θ_{12} is non zero

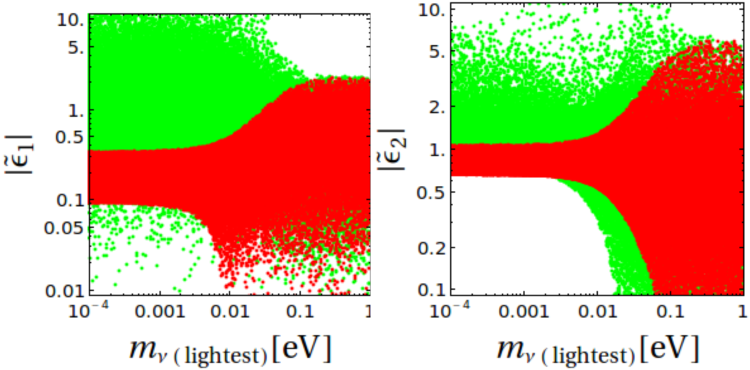
$$\mathcal{M}_{\nu f} = \begin{pmatrix} x & a & \tilde{\epsilon}_1 c \\ a & b & \tilde{\epsilon}_2 c \\ \tilde{\epsilon}_1 c & \tilde{\epsilon}_2 c & c \end{pmatrix} ,$$

We find the conditions under which $\tilde{\epsilon}_{1,2}$ are small

Approximately $\mu - \tau$ Symmetric $\mathbf{M_l}$ and $\mathbf{M_\nu}$

$$\tilde{\epsilon}_1 = \frac{c_{13}c_{12}s_{12}(m_1 - m_2 e^{-i\alpha_2})(-s_{13}c_{23}f + s_{23})}{c_{23}^2 g_+ - \cos 2\theta_{23}(m_2 e^{-i\alpha_2} c_{12}^2 + m_1 s_{12}^2) + \sin 2\theta_{23} c_{12}s_{12}s_{13}e^{-i\delta}(m_1 - m_2 e^{-i\alpha_2})},$$

$$\tilde{\epsilon}_2 = \frac{c_{23}s_{23}g_- + \cos 2\theta_{23} c_{12}s_{12}s_{13}e^{-i\delta}(m_1 - m_2 e^{-i\alpha_2})}{c_{23}^2 g_+ - \cos 2\theta_{23}(m_2 e^{-i\alpha_2} c_{12}^2 + m_1 s_{12}^2) + \sin 2\theta_{23} c_{12}s_{12}s_{13}e^{-i\delta}(m_1 - m_2 e^{-i\alpha_2})}.$$



Approximately $\mu - \tau$ Symmetric $\mathbf{M_l}$ and $\mathbf{M_\nu}$

- For NH and IH, $|\tilde{\epsilon}_1|, |\tilde{\epsilon}_2| \ll 0.2$ if $m_0 > 0.04$ eV. We cannot regard $\tilde{\mathbf{S}}_2$ as approximate symmetry of $\mathcal{M}_{\nu f}$ for the hierarchical neutrinos.
- For QD, $\tilde{\mathbf{S}}_2$ and $\mathbf{S}_2$ at the fundamental level can be an approximately good symmetry.
- Diagonalization of $\mathbf{M}_{\nu f}$ gives

$$
\begin{aligned}
\tan 2\theta_{23} &\approx \frac{2c\tilde{\epsilon}_2}{b - c}, \\
\tan 2\theta_{12} &\approx \frac{2(ac_{23} + c\tilde{\epsilon}_1 s_{23})}{m_2 - x}, \\
\tan 2\theta_{13} &\approx \frac{2(c\tilde{\epsilon}_1 c_{23} - a s_{23})}{m_3 - x},
\end{aligned}
$$

Approximately $\mu - \tau$ Symmetric $\mathbf{M_l}$ and $\mathbf{M_\nu}$

$$
\begin{aligned}
\mathbf{m_3} &\approx \frac{1}{2}\left(\mathbf{b}+\mathbf{c}-\frac{\mathbf{b}-\mathbf{c}}{\cos 2\theta_{23}}\right), \\
\mathbf{m_2} &\approx \frac{1}{2}\left(\mathbf{b}+\mathbf{c}+\frac{\mathbf{b}-\mathbf{c}}{\cos 2\theta_{23}}\right), \\
\mathbf{m_1} &\approx \frac{1}{2}\left(\mathbf{x}+\mathbf{m_2}+\frac{\mathbf{x}-\mathbf{m_2}}{\cos 2\theta_{12}}\right).
\end{aligned}
$$

- Large atmospheric mixing is consistent with small $\tilde{\epsilon}_2$ for $\mathbf{b} \approx \mathbf{c} \gg \tilde{\epsilon}_2$ which corresponds to $\mathbf{m_2} \sim \mathbf{b} + \mathbf{c}\tilde{\epsilon}_2$, $\mathbf{m_3} \sim \mathbf{b} - \mathbf{c}\tilde{\epsilon}_2$. $\mathbf{m_1}$ is then required to be degenerate if both solar and atmospheric neutrino scales are to be reproduced. For $\mathbf{c} \ll \mathbf{b}$ or $\mathbf{b} \ll \mathbf{c}$, $\tilde{\epsilon}_2$ is $\mathcal{O}(1)$ and needs a large μ-τ breaking.

Approximately $\mu - \tau$ Symmetric $\mathbf{M_l}$ and $\mathbf{M_\nu}$

- We fit parameters in two different ways. We minimize relevant χ^2 by restricting $\tilde{\epsilon}_{1,2}$ to be $\leq \mathbf{0.1}$.

$$\mathcal{M}_{\nu f} = \mathbf{0.076803} \;\; \mathbf{eV} \begin{pmatrix} 0.8814 & -0.02279 & -0.02223 \\ -0.02279 & 0.9624 & 0.1 \\ -0.02223 & 0.1 & 1 \end{pmatrix}$$

$$(\mathbf{m_1, m_2, m_3}) = (\mathbf{0.06716, 0.06771, 0.08355}) \quad \mathbf{eV}$$

We get correct central values of $\theta_{23}, \theta_{13}, \theta_{12}$ and $\Delta_\odot/\Delta_A$.

Performing same fit without putting restrictions on $\tilde{\epsilon}_{1,2}$ leads to

$$\mathcal{M}_{\nu f} = \mathbf{0.03189} \;\; \mathbf{eV} \begin{pmatrix} 0.03992 & 0.2954 & 0.05802 \\ 0.2954 & 0.6988 & 0.6615 \\ 0.05802 & 0.6615 & 1 \end{pmatrix}$$

$$(\mathbf{m_1, m_2, m_3}) = (\mathbf{0.00388, 0.00948, 0.04985}) \quad \mathbf{eV}$$

large symmetry breaking required for values of observables near central values

Approximately $\mu - \tau$ Symmetric Lagrangian

Lagrangian is approx μ-τ symmetric then $\mathbf{M_l}$ would also display (approx) μ-τ symmetry but $\mathbf{M}_\nu$ may or may not.

- If neutrino masses result from the type-II seesaw mechanism with direct coupling of a triplet Higgs to neutrinos then like $\mathbf{M_l}$, $\mathbf{M}_\nu$ will display approx μ-τ symmetry
- If neutrino masses are from type-I seesaw mechanism we regard $\mathbf{m_D}$ and $\mathbf{M_R}$ as approx μ-τ symmetric. $\mathbf{M}_\nu \approx -\mathbf{m_D} \mathbf{M_R}^{-1} \mathbf{m_D^T}$ may have large breaking of μ-τ symmetry even when $\mathbf{m_D}, \mathbf{M_R}$ are approximately μ-τ symmetric. Thus in type-I seesaw mechanism, even the hierarchical neutrino spectrum can be consistent with the approximately μ-τ symmetric Lagrangian
- Thus, hierarchical neutrinos can be consistent with approx μ-τ symmetry

Approximately $\mu - \tau$ Symmetric Lagrangian

If $\mathbf{M_l}$ and $\mathbf{M_R}$ are μ-τ sym and breaking occur in $\mathbf{M_D}$

$$\mathbf{m_D} = \begin{pmatrix} x_D & a_D(1-\epsilon_{1D}) & a_D(1+\epsilon_{1D}) \\ a_D(1-\epsilon_{1D}) & b_D(1-\epsilon_{2D}) & c_D \\ a_D(1+\epsilon_{1D}) & c_D & b_D(1+\epsilon_{2D}) \end{pmatrix},$$

- if the EV of $\mathbf{m_D}$ and $\mathbf{M_R}$ are hierarchical [11] then resulting $\mathbf{M_\nu}$ has large breaking of μ-τ symmetry

$$\mathcal{M}_{\nu f} \approx \tilde{m}_D^T \,\text{diag.}(M_1^{-1}, M_2^{-1}, M_3^{-1})\, \tilde{m}_D$$

$$\tilde{m}_D \equiv R_{23}^T(\pi/4)\, m_D\, R_{23}(\pi/4)$$

$$\tilde{\epsilon}_1 \approx \frac{\sqrt{2}a_D(\epsilon_{2D}b_D M_1 M_3 + \epsilon_{1D}M_2(b_D M_1 - c_D M_1 + M_3 x_D))}{(b_D - c_D)^2 M_1 M_2},$$

$$\tilde{\epsilon}_2 \approx \frac{2\epsilon_{1D}M_2M_3a_D^2 + \epsilon_{2D}b_D M_1(c_D(M_3 - M_2) + b_D(M_2 + M_3))}{(b_D - c_D)^2 M_1 M_2}.$$

[11] A. S. Joshipura, Eur. Phys. J. C 53,77 (2008)

Approximately $\mu - \tau$ Symmetric Lagrangian

For $\mathbf{m_1} \ll \mathbf{m_2} \ll \mathbf{m_3}$ requires small $\tilde{\epsilon}_1$ and relatively large $\tilde{\epsilon}_2$

- can be reconciled with a small breaking $\epsilon_{1D,2D}$ at the fundamental level

  For $\mathbf{m_{1D}} \ll \mathbf{m_{2D}} \ll \mathbf{m_{3D}}$

$$\frac{\tilde{\epsilon}_1}{\tilde{\epsilon}_2} \approx \frac{\sqrt{2m_{1D}/m_{2D}}}{1 + \frac{M_2}{M_3}\frac{m_{2D}}{m_{3D}}}, \tilde{\epsilon}_2 \approx \frac{\epsilon_{2D}}{2}\left(1 + \frac{m_{2D}}{m_{3D}}\right)\left(1 + \frac{m_{2D}M_3}{m_{3D}M_2}\right).$$

- Strong RH hierarchy $\frac{M_2}{M_3} \ll \frac{m_{2D}}{m_{3D}}$ and hierarchical $\mathbf{m_{iD}}$ lead to enhancement in $\tilde{\epsilon}_2$ compared to the basic parameter ϵ_{2D} and the ratio $\frac{\tilde{\epsilon}_1}{\tilde{\epsilon}_2}$ remains small as required.

- A class of symmetry can be obtained by combining $\mathbf{Z_2} \times \mathbf{Z_2}$ symmetry with CP (generalized $\mathbf{Z_2} \times \mathbf{Z_2}$)
- Imposition of such symmetry lead to

$$\text{I)}\ \sin^2\theta_{23} = \frac{1}{2}, \sin^2\theta_{12} = \frac{1}{3}(1 + \tan^2\theta_{13})$$

$$\text{II)}\ \sin^2\theta_{23} = \frac{1}{2}, \sin^2\theta_{12} = \frac{1}{3}(1 - 2\tan^2\theta_{13}), \delta = \pi/2$$

- Case II can be obtained by extending the $\mathbf{A_4}$ model either through SU(2) triplet or singlet field both triplet under $\mathbf{A_4}$. The model leads to prediction for absolute neutrino mass as a function of reactor angle

THANK YOU