A survey of B Physics
in Covariant Light Front model

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Flavor Physics in B Mesons

Introduction

- B decays provide a sensitive probe of the physics of quark mixing, described by the unitary Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix in the Standard Model (SM).
- The mixing of the weak and mass eigenstates of the quarks provides a rich phenomenology and gives a viable mechanism for the non-conservation of CP symmetry in the decays of certain hadrons.
- CP asymmetries in B decays can be large and allow a determination of the magnitude of the irreducible phase in the CKM matrix.
- The study of B decays allows direct measurement of the magnitudes of the elements $|V_{\mathrm{ub}}|$ and $|V_{\mathrm{cb}}|$ of the CKM matrix.
- B decays provide a laboratory for testing our understanding of QCD.

- The scale of the short-distance physics (e.g. weak b quark decay) is in the perturbative regime of QCD while the formation of final state hadrons and the binding of the b quark to the valence anti-quark is non-perturbative.
- The Operator Product Expansion (OPE) with Heavy Quark Symmetry (HQS) allow perturbative calculations to be combined with non-perturbative matrix elements, and provide relationships in the non-perturbative matrix elements contributing to different processes.
- Powerful theoretical tools have been developed to systematically address the disparate scales.
- This allows some non-perturbative quantities to be determined experimentally, and leads to the interplay between experiment and theory that has characterized this area of research in recent years.

The main actor is B-meson system, we distinguish between charged and neutral B mesons, which are characterized by the following valence-quark contents:

$$\begin{array}{llll} B^+ \sim u\bar{b}, & B_c^+ \sim c\bar{b}, & B_d^0 \sim d\bar{b}, & B_s^0 \sim s\bar{b}, \\ B^- \sim \bar{u}b, & B_c^- \sim \bar{c}b, & \bar{B}_d^0 \sim \bar{d}b, & \bar{B}_s^0 \sim \bar{s}b. \end{array}$$

Besides the charged B mesons, their neutral counterparts B_q $(q \in \{d, s\})$ show the phenomenon of B_q^0–$\bar{B}_q^0$ mixing.

CP Violation in Standard Model

The framework of the Standard Model of electroweak interactions is based on the spontaneously broken gauge group

$$SU(2)_L \times U(1)_Y \xrightarrow{\text{SSB}} U(1)_{em}$$

CP-violating effects may originate from the charged-current interactions of quarks, having the structure

$$D \to UW^-$$

Here $D \in \{d, s, b\}$ and $U \in \{u, c, t\}$ denote down- and up-type quark flavors, respectively, whereas the W^- is the usual $SU(2)_L$ gauge boson.

Phenomenological point of view: it is convenient to collect the generic "coupling strengths" V_{UD} of the charged-current processes in in the form of the following matrix:

$$\hat{V}_{\mathrm{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

which is referred to as the Cabibbo–Kobayashi–Maskawa (CKM) matrix.

Theoretical point of view: this matrix connects the electroweak states (d', s', b') of the down, strange and bottom quarks with their mass eigenstates (d, s, b) through the following unitary transformation

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

$\hat{V}_{\mathrm{CKM}}$ is actually a *unitary* matrix.
This feature ensures the absence of flavor-changing neutral-current (FCNC) processes at the tree level in the SM, and is hence at the basis of the famous Glashow–Iliopoulos–Maiani (GIM) mechanism.

Expressed in the non-leptonic charged-current interaction Lagrangian in terms of the mass eigenstates in

$$\mathcal{L}_{\text{int}}^{\text{CC}} = -\frac{g_2}{\sqrt{2}} \begin{pmatrix} \bar{u}_L, & \bar{c}_L, & \bar{t}_L \end{pmatrix} \gamma^\mu \hat{V}_{\text{CKM}} \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix} W_\mu^\dagger + \text{ h.c.},$$

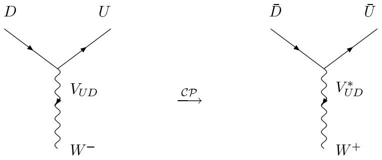
where g_2 is the gauge group $SU(2)_L$, and the $W_\mu^{(\dagger)}$ field corresponds to the charged W bosons.
The elements of the CKM matrix describe the generic strengths of the associated charged-current processes.

The corresponding CP transformation involves the replacement

$$\mathrm{V_{UD}} \xrightarrow{\mathrm{CP}} \mathrm{V^*_{UD}}$$

CP violation could in principle be accommodated in the SM through complex phases in the CKM matrix. May actually have physical complex phases in that matrix.

Figure: $\mathrm{D} \to \mathrm{U}W^-$ vertex and its CP conjugate



Phases of CKM Matrix

Freedom to redefine the up- and down-type quark fields in the following manner:

$$U \to \exp(i\xi_U)U, \quad D \to \exp(i\xi_D)D.$$

Performing the transformations, the invariance of the charged-current interaction Lagrangian implies the phase transformations of the CKM matrix elements:

$$V_{UD} \to \exp(i\xi_U)V_{UD}\exp(-i\xi_D).$$

Using these transformations to eliminate unphysical phases, the parametrization of the general $N \times N$ quark-mixing matrix involves the parameters:

$$\underbrace{\frac{1}{2}N(N-1)}_{\text{Euler angles}} + \underbrace{\frac{1}{2}(N-1)(N-2)}_{\text{complex phases}} = (N-1)^2.$$

N is the number of fermion generations.

The case of $N = 2$ generations, only one rotation angle – the Cabibbo angle θ_C – is required for the parametrization of the 2×2 quark-mixing matrix, which can be written in the following form:

$$\hat{V}_C = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix},$$

where $\sin\theta_C = 0.22$ can be determined from $K \to \pi\ell\bar{\nu}$ decays.

The case of $N = 3$ generations, the parametrization involves three Euler-type angles and a single *complex* phase. This complex phase is to accommodate CP violation in the SM. In the **Standard Parametrization** by PDG, the three-generation CKM matrix takes the following form:

$$\hat{V}_{\text{CKM}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix}$$

where $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$. The advantage of this parametrization is that the generation labels $i, j = 1, 2, 3$ are introduced in such a manner that the mixing between two chosen generations vanishes if the corresponding mixing angle θ_{ij} is set to zero. In particular, for $\theta_{23} = \theta_{13} = 0$, the third generation decouples, and the 2×2 submatrix describing the mixing between the first and second generations.

Words of wisdom

Physical observables, for instance CP-violating asymmetries, *cannot* depend on the chosen parametrization of the CKM matrix, i.e. Physics has to be invariant under the phase transformations.

- To accommodate CP violation within the framework of the SM through a complex phase in the CKM matrix, at least three generations are required. However, this feature is not sufficient for observable CP-violating effects.
- Another conditions have to be satisfied:

$$(m_t^2-m_c^2)(m_t^2-m_u^2)(m_c^2-m_u^2)(m_b^2-m_s^2)(m_b^2-m_d^2)(m_s^2-m_d^2)\times J_{CP} \neq 0$$

where

$$J_{CP} = |\mathrm{Im}(V_{i\alpha}V_{j\beta}V_{i\beta}^{*}V_{j\alpha}^{*})| \quad (i \neq j,\ \alpha \neq \beta)$$

The “Jarlskog parameter” J_{CP}, can be interpreted as a measure of the strength of CP violation in the SM.

Notes on CPV requirements

- The mass factors are related to the fact that the CP-violating phase of the CKM matrix could be eliminated through an appropriate unitary transformation of the quark fields if any two quarks with the same charge had the same mass.
- The origin of CP violation is closely related to the "flavor problem" in elementary particle physics, and cannot be understood in a deeper way, unless we have fundamental insights into the hierarchy of quark masses and the number of fermion generations.
- In the standard parametrization

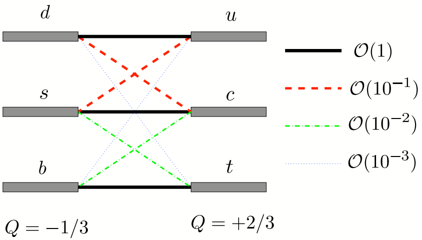
$$J_{CP} = s_{12}s_{13}s_{23}c_{12}c_{23}c_{13}^2 \sin\delta_{13}.$$

- The experimental data on the CKM parameters implies $J_{CP} = \mathcal{O}(10^{-5})$. CP-violating phenomena are hard to observe.
- However, new complex couplings are typically present in scenarios for NP. Such additional sources for CP violation could be detected through flavor experiments.

From experimental information, assuming three generations and CKM unitarity condition, the $|V_{ij}|$:

$$|\hat{V}_{\mathrm{CKM}}| = \begin{pmatrix} 0.9739\text{–}0.9751 & 0.221\text{–}0.227 & 0.0029\text{–}0.0045 \\ 0.221\text{–}0.227 & 0.9730\text{–}0.9744 & 0.039\text{–}0.044 \\ 0.0048\text{–}0.014 & 0.037\text{–}0.043 & 0.9990\text{–}0.9992 \end{pmatrix}$$

Figure: Hierarchy of the quark transitions mediated through charged-current processes.



Phenomenological parametrization

Introducing a set of new parameters, λ, A, ρ and η, imposing the following relations:

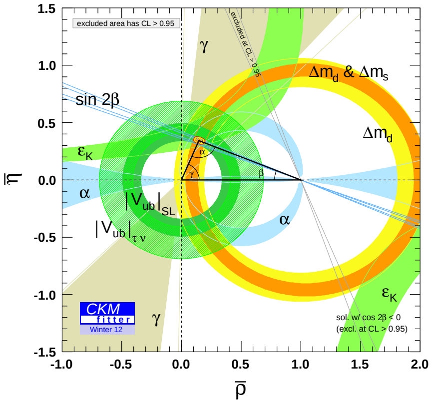
$$s_{12} \equiv \lambda = 0.22, \quad s_{23} \equiv A\lambda^2, \quad s_{13}e^{-i\delta_{13}} \equiv A\lambda^3(\rho - i\eta).$$

and neglecting terms of $\mathcal{O}(\lambda^4)$, applying Standard Parametrization, obtaining the famous "Wolfenstein parametrization"

$$\hat{V}_{\mathrm{CKM}} = \begin{pmatrix} 1 - \frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4),$$

it makes the hierarchical structure of the CKM matrix very transparent and is an important tool for phenomenological considerations

$$\bar{\rho} \equiv \rho\left(1 - \frac{1}{2}\lambda^2\right), \quad \bar{\eta} \equiv \eta\left(1 - \frac{1}{2}\lambda^2\right)$$



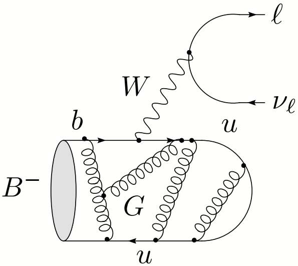
Decays of B mesons

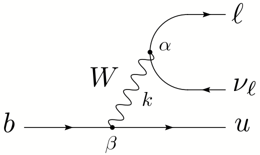
- B-meson system consists of charged and neutral B mesons, which are characterized by the valence quark contents.
- The characteristic feature of the neutral B_q $(q \in \{d, s\})$ mesons is the phenomenon of B_q^0–$\bar{B}_q^0$ mixing.
- Types of B meson decays are distinguished between leptonic, semileptonic and non-leptonic transitions.

Leptonic decays

$$T_{fi} = -\frac{g_2^2}{8} V_{ub} \underbrace{[\bar{u}_\ell \gamma^\alpha (1-\gamma_5) v_\nu]}_{\text{Dirac spinors}} \left[\frac{g_{\alpha\beta}}{k^2 - M_W^2}\right] \underbrace{\langle 0|\bar{u}\gamma^\beta(1-\gamma_5)b|B^-\rangle}_{\text{hadronic ME}},$$

Figure: Feynman diagrams contributing to the leptonic decay $B^- \to \ell \bar{\nu}_\ell$.





For $k^2 = M_B^2 \ll M_W^2$

$$\frac{g_{\alpha\beta}}{k^2 - M_W^2} \quad \longrightarrow \quad -\frac{g_{\alpha\beta}}{M_W^2} \equiv -\left(\frac{8G_F}{\sqrt{2}g_2^2}\right) g_{\alpha\beta}$$

where G_F is Fermi's constant. "Integrating out" the W boson

$$T_{fi} = \frac{G_F}{\sqrt{2}} V_{ub} \left[\bar{u}_\ell \gamma^\alpha (1-\gamma_5) v_\nu\right] \langle 0|\bar{u}\gamma_\alpha(1-\gamma_5)b|B^-\rangle.$$

All the hadronic physics is encoded in the *hadronic matrix element*

$$\langle 0|\bar{u}\gamma_\alpha(1-\gamma_5)b|B^-\rangle,$$

i.e. there are no other strong-interaction QCD effects. Since the B^- meson is a pseudoscalar particle

$$\langle 0|\overline{u}\gamma_\alpha b|B^-\rangle = 0,$$

and may write

$$\langle 0|\bar{u}\gamma_\alpha\gamma_5 b|B^-(q)\rangle = i f_B q_\alpha$$

Performing the corresponding phase-space integrations, the following decay rate is obtained:

$$\Gamma(B^- \to \ell\bar{\nu}_\ell) = \frac{G_F^2}{8\pi} M_B m_\ell^2 \left(1 - \frac{m_\ell^2}{M_B^2}\right)^2 f_B^2 |V_{ub}|^2$$

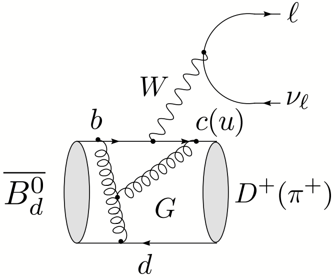
Remarks:

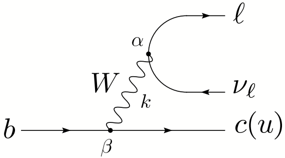
- The tiny value of $|V_{ub}| \propto \lambda^3$ and a helicity-suppression mechanism: very small branching ratios of $\mathcal{O}(10^{-10})$ and $\mathcal{O}(10^{-7})$ for $\ell = e$ and $\ell = \mu$
- The helicity suppression is not effective for $\ell = \tau$
- Knowledge of $|V_{ub}|$ is needed to extract f_B, to tests the non-perturbative calculations of this important parameter.

Semileptonic decays

$$T_{fi} = -\frac{g_2^2}{8} V_{cb} \underbrace{[\bar{u}_\ell \gamma^\alpha (1-\gamma_5) v_\nu]}_{\text{Dirac spinors}} \left[\frac{g_{\alpha\beta}}{k^2 - M_W^2}\right] \underbrace{\langle D^+ | \bar{c} \gamma^\beta (1-\gamma_5) b | \bar{B}_d^0 \rangle}_{\text{hadronic ME}} .$$

Figure: Feynman diagrams contributing to semileptonic $\bar{B}_d^0 \to D^+(\pi^+)\ell\bar{\nu}_\ell$ decays.





For $k^2 \sim M_B^2 \ll M_W^2$

$$T_{fi} = \frac{G_F}{\sqrt{2}} V_{cb} \left[\bar{u}_\ell \gamma^\alpha (1-\gamma_5) v_\nu\right] \langle D^+ | \bar{c}\gamma_\alpha (1-\gamma_5) b | \bar{B}_d^0 \rangle,$$

where *all* the hadronic physics is encoded in the hadronic matrix element

$$\langle D^+ | \bar{c}\gamma_\alpha (1-\gamma_5) b | \bar{B}_d^0 \rangle,$$

i.e. there are *no* other QCD effects. $\bar{B}_d^0$ and D^+ are pseudoscalar mesons

$$\langle D^+ | \bar{c}\gamma_\alpha \gamma_5 b | \bar{B}_d^0 \rangle = 0,$$

and may write

$$\begin{aligned} \langle D^+(k) | \bar{c}\gamma_\alpha b | \bar{B}_d^0(p) \rangle &= F_1(q^2) \left[(p+k)_\alpha - \left(\frac{M_B^2 - M_D^2}{q^2} \right) q_\alpha \right] \\ &+ F_0(q^2) \left(\frac{M_B^2 - M_D^2}{q^2} \right) q_\alpha, \end{aligned}$$

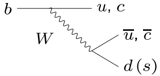
where $q \equiv p - k$, $F_{1,0}(q^2)$ is the *form factors* of the $\bar{B} \to D$ transitions. To calculate these parameters, non-perturbative techniques (QCD sum rules, lattice, etc.) are required.

Non-Leptonic decays

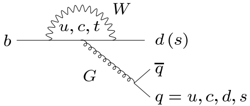
- There are two kinds of topologies contributing to such decays: tree-diagram-like and "penguin" topologies.
- The latter consist of gluonic (QCD) and electroweak (EW) penguins.
- Depending on the flavour content of their final states, classify the decays $b \to q_1\, \bar{q}_2\, d\,(s)$ as follows:
  - $q_1 \neq q_2 \in \{u, c\}$: *only* tree diagrams contribute.
  - $q_1 = q_2 \in \{u, c\}$: tree *and* penguin diagrams contribute.
  - $q_1 = q_2 \in \{d, s\}$: *only* penguin diagrams contribute.

Figure: Feynman diagrams of the topologies characterizing non-leptonic B decays: trees (a), QCD penguins (b), and electroweak penguins (c).

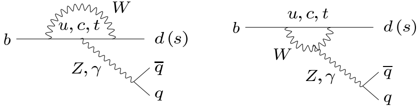
(a)



(b)



(c)



Effective Hamiltonians

Transition matrix elements:

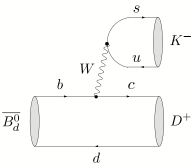
$$\langle f|\mathcal{H}_{\text{eff}}|i\rangle = \frac{G_F}{\sqrt{2}}\lambda_{\text{CKM}}\sum_k C_k(\mu)\langle f|Q_k(\mu)|i\rangle \,.$$

- $C_k(\mu)$ ("Wilson coefficient functions") perturbative quantities.
- $\langle f|Q_k(\mu)|i\rangle$ ("hadronic matrix elements") non-perturbative quantities.
- The Q_k are local operators, generated by electroweak interactions and QCD.
- The Wilson coefficients $C_k(\mu)$ can be considered as scale-dependent couplings related to the vertices described by the Q_k.

$$-\frac{g_2^2}{8}V_{us}^*V_{cb}\,[\bar{s}\gamma^\nu(1-\gamma_5)u]\left[\frac{g_{\nu\mu}}{k^2-M_W^2}\right][\bar{c}\gamma^\mu(1-\gamma_5)b]\,.$$

$$\begin{aligned}\mathcal{H}_{\text{eff}} &= \frac{G_F}{\sqrt{2}}V_{us}^*V_{cb}\,[\bar{s}_\alpha\gamma_\mu(1-\gamma_5)u_\alpha]\,[\bar{c}_\beta\gamma^\mu(1-\gamma_5)b_\beta] \\ &= \frac{G_F}{\sqrt{2}}V_{us}^*V_{cb}(\bar{s}_\alpha u_\alpha)_{V-A}(\bar{c}_\beta b_\beta)_{V-A} \equiv \frac{G_F}{\sqrt{2}}V_{us}^*V_{cb}O_2\,,\end{aligned}$$

Figure: Feynman diagrams contributing to the non-leptonic $\bar{B}_d^0 \to D^+K^-$ decay.



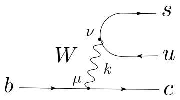


Figure: The description of the $b \to d\bar{u}s$ process through the four-quark operator O_2 in the effective theory after the W boson has been integrated out.

$$O_1 \equiv [\bar{s}_\alpha \gamma_\mu (1-\gamma_5) u_\beta] \, [\bar{c}_\beta \gamma^\mu (1-\gamma_5) b_\alpha] \, .$$

$$\mathcal{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} V_{us}^* V_{cb} \, [C_1(\mu) O_1 + C_2(\mu) O_2] \, .$$

Figure: Factorizable QCD corrections in the full and effective theories

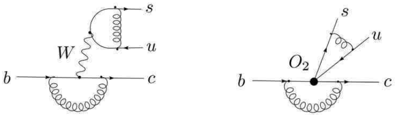
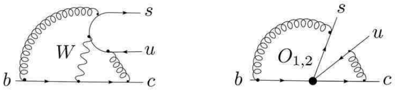


Figure: Non-Factorizable QCD corrections in the full and effective theories



Full Effective Hamiltonians

$$\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}}\Big\{ \sum_{q=u,c} V_{qb}V^*_{qD}\left[C_1(\mu)O_1^q(\mu) + C_2(\mu)O_2^q(\mu)\right]$$
$$-V_{tb}V^*_{tD}\left[\sum_{i=3}^{10} C_i(\mu)O_i(\mu)\right]\Big\} + \mathrm{H.c.}$$

- Current–current operators:

$$\begin{aligned} Q_1^{jr} &= (\bar{r}_\alpha j_\beta)_{V-A}(\bar{j}_\beta b_\alpha)_{V-A} \\ Q_2^{jr} &= (\bar{r}_\alpha j_\alpha)_{V-A}(\bar{j}_\beta b_\beta)_{V-A}. \end{aligned}$$

- QCD penguin operators:

$$\begin{aligned} Q_3^r &= (\bar{r}_\alpha b_\alpha)_{V-A} \textstyle\sum_{q'}(\bar{q}'_\beta q'_\beta)_{V-A} \\ Q_4^r &= (\bar{r}_\alpha b_\beta)_{V-A} \textstyle\sum_{q'}(\bar{q}'_\beta q'_\alpha)_{V-A} \\ Q_5^r &= (\bar{r}_\alpha b_\alpha)_{V-A} \textstyle\sum_{q'}(\bar{q}'_\beta q'_\beta)_{V+A} \\ Q_6^r &= (\bar{r}_\alpha b_\beta)_{V-A} \textstyle\sum_{q'}(\bar{q}'_\beta q'_\alpha)_{V+A}. \end{aligned}$$

- EW penguin operators (the $e_{q'}$ denote the electrical quark charges):

$$
\begin{aligned}
Q_7^r &= \tfrac{3}{2}(\bar{r}_\alpha b_\alpha)_{V-A} \sum_{q'} e_{q'} (\bar{q}'_\beta q'_\beta)_{V+A} \\
Q_8^r &= \tfrac{3}{2}(\bar{r}_\alpha b_\beta)_{V-A} \sum_{q'} e_{q'} (\bar{q}'_\beta q'_\alpha)_{V+A} \\
Q_9^r &= \tfrac{3}{2}(\bar{r}_\alpha b_\alpha)_{V-A} \sum_{q'} e_{q'} (\bar{q}'_\beta q'_\beta)_{V-A} \\
Q_{10}^r &= \tfrac{3}{2}(\bar{r}_\alpha b_\beta)_{V-A} \sum_{q'} e_{q'} (\bar{q}'_\beta q'_\alpha)_{V-A}.
\end{aligned}
$$

- At a renormalization scale $\mu = \mathcal{O}(m_b)$, $C_1(\mu) = \mathcal{O}(10^{-1})$ and $C_2(\mu) = \mathcal{O}(1)$, whereas the penguin operators are $\mathcal{O}(10^{-2})$.

Covariant Light Front model

Background

- The light-front quark model is the only relativistic quark model in which a consistent and fully relativistic treatment of quark spins and the center-of-mass motion can be carried out.
- The light-front wave function is manifestly Lorentz invariant as it is expressed in terms of the momentum fraction variables in analog to the parton distributions in the infinite momentum frame.
- Suitable to study hadronic form factors, especially as the recoil momentum increases (a decreasing q^2). At $q^2 = 0$ the final-state meson could be highly relativistic.

Issue

- Picking up a specific Lorentz frame (e.g. the purely longitudinal frame $q_\perp = 0$, or the purely transverse frame $q^+ = q^0 + q^3 = 0$) and Calculating a particular component (the "plus" component) of the associated current matrix element will miss the zero-mode contributions and render the matrix element non-covariant.
- The lack of relativistic covariance makes the results not unique and may even cause some inconsistencies. Some ambiguities and even some inconsistencies in extracting the physical quantities.
- For instance: in the $q_\perp = 0$ frame the so-called Z-diagram contributions must be incorporated in the form-factor calculations in order to maintain covariance.

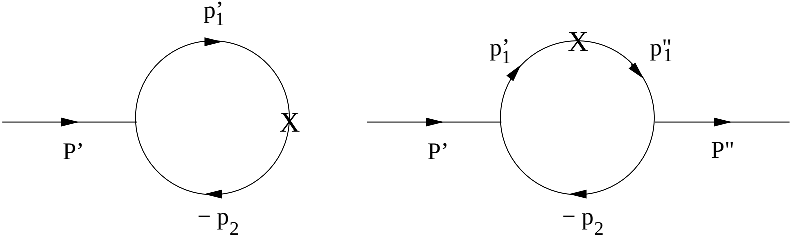
Workaround

- The starting point of the covariant approach is to consider the corresponding covariant Feynman amplitudes in meson transitions.
- One can pass to the light-front approach by using the light-front decomposition of the internal momentum in covariant Feynman momentum loop integrals and integrating out the $p^- = p^0 - p^3$ component.
- Then apply some well-studied vertex functions in the conventional light-front approach after p^- integration.
- In going from the manifestly covariant Feynman integral to the light-front one makes it non-covariant as it receives additional spurious contributions proportional to the lightlike vector $\tilde{\omega}^\mu = (1, 0, 0, -1)$. This spurious contribution is cancelled after correctly performing the integration by the inclusion of the zero mode contribution.

- The zero mode contributions can be interpreted as residues of virtual pair creation processes in the $q^+ \to 0$ limit.
- The calculation of the zero mode contribution is obtained in a frame where the momentum transfer q^+ vanishes.
- Form factors are known only for spacelike momentum transfer $q^2 = -q_\perp^2 \leqslant 0$. One needs to analytically continue them to the timelike region, where the physical decay processes are relevant.
- There are some theoretical constraints implied by heavy quark symmetry (HQS) in the case of heavy-to-heavy transitions and heavy-to-vacuum decays.
- Under HQS the number of the independent form factors is reduced and they are related to some universal Isgur-Wise (IW) functions.

Formalism

Figure: Feynman diagrams for meson decay (left), and meson transition amplitudes (right), where $P'^{(\prime\prime)}$ is the incoming (outgoing) meson momentum, $p_1'^{(\prime\prime)}$ is the quark momentum, p_2 is the anti-quark momentum and X denotes the corresponding $V - A$ current vertex.



The incoming (outgoing) meson has the momentum $P'^{(\prime\prime)} = p_1'^{(\prime\prime)} + p_2$, where $p_1'^{(\prime\prime)}$ and p_2 are the momenta of the off-shell quark and antiquark, respectively, with masses $m_1'^{(\prime\prime)}$ and m_2.

These momenta can be expressed in terms of the internal variables $(x_i, p'_\perp)$,

$$p'^+_{1,2} = x_{1,2} P'^+, \qquad p'_{1,2\perp} = x_{1,2} P'_\perp \pm p'_\perp,$$

with $x_1 + x_2 = 1$. Note that $P' = (P'^+, P'^-, P'_\perp)$, where $P'^\pm = P'^0 \pm P'^3$, so that $P'^2 = P'^+ P'^- - P'^2_\perp$.
In the covariant light-front approach, total four momentum is conserved at each vertex where quarks and antiquarks are off-shell. It is useful to define some internal quantities for on-shell quarks:

$$\begin{aligned} M'^2_0 &= (e'_1 + e_2)^2 = \frac{p'^2_\perp + m'^2_1}{x_1} + \frac{p'^2_\perp + m^2_2}{x_2}, \quad \widetilde{M}'_0 = \sqrt{M'^2_0 - (m'_1 - m_2)^2}, \\ e^{(\prime)}_i &= \sqrt{m^{(\prime)2}_i + p'^2_\perp + p'^2_z}, \quad p'_z = \frac{x_2 M'_0}{2} - \frac{m^2_2 + p'^2_\perp}{2 x_2 M'_0}. \end{aligned}$$

Here M'^2_0 can be interpreted as the kinetic invariant mass squared of the incoming $q\bar{q}$ system, and e_i the energy of the quark i.

Table: Feynman rules for the vertices ($i\Gamma'_M$) of the incoming mesons-quark-antiquark, where p'_1 and p_2 are the quark and antiquark momenta, respectively. Under the contour integrals, H'_M and W'_M are reduced to h'_M and w'_M, respectively. For outgoing mesons, we shall use $i(\gamma_0 \Gamma'^{\dagger}_M \gamma_0)$ for the corresponding vertices.

$M\ ({}^{2S+1}L_J)$	$i\Gamma'_M$
pseudoscalar $({}^1S_0)$	$H'_P \gamma_5$
vector $({}^3S_1)$	$iH'_V[\gamma_\mu - \frac{1}{W'_V}(p'_1 - p_2)_\mu]$
scalar $({}^3P_0)$	$-iH'_S$
axial $({}^3P_1)$	$-iH'_{{}^3A}[\gamma_\mu + \frac{1}{W'_{{}^3A}}(p'_1 - p_2)_\mu]\gamma_5$
axial $({}^1P_1)$	$-iH'_{{}^1A}[\frac{1}{W'_{{}^1A}}(p'_1 - p_2)_\mu]\gamma_5$
tensor $({}^3P_2)$	$i\frac{1}{2}H'_T[\gamma_\mu - \frac{1}{W'_V}(p'_1 - p_2)_\mu](p'_1 - p_2)_\nu$

The explicit forms of h'_M and w'_M are given by

$$
\begin{aligned}
h'_P &= h'_V = (M'^2 - M_0'^2)\sqrt{\frac{x_1 x_2}{N_c}}\frac{1}{\sqrt{2}\widetilde{M}'_0}\varphi', \\
h'_S &= \sqrt{\frac{2}{3}}h'_{^3A} = (M'^2 - M_0'^2)\sqrt{\frac{x_1 x_2}{N_c}}\frac{1}{\sqrt{2}\widetilde{M}'_0}\frac{\widetilde{M}_0'^2}{2\sqrt{3}M'_0}\varphi'_p, \\
h'_{^1A} &= h'_T = (M^{2\prime} - M_0'^2)\sqrt{\frac{x_1 x_2}{N_c}}\frac{1}{\sqrt{2}\widetilde{M}'_0}\varphi'_p, \\
w'_V &= M'_0 + m'_1 + m_2, \quad w'_{^3A} = \frac{\widetilde{M}_0'^2}{m'_1 - m_2}, \quad w'_{^1A} = 2,
\end{aligned}
$$

where φ' and φ'_p are the light-front momentum distribution amplitudes for s-wave and p-wave mesons.

Phenomenological light-front wave functions: Gaussian-type wave function.

$$
\begin{aligned}
\varphi' &= \varphi'(x_2, p'_\perp) = 4\left(\frac{\pi}{\beta'^2}\right)^{\frac{3}{4}} \sqrt{\frac{dp'_z}{dx_2}}\, \exp\left(-\frac{p'^2_z + p'^2_\perp}{2\beta'^2}\right), \\
\varphi'_p &= \varphi'_p(x_2, p'_\perp) = \sqrt{\frac{2}{\beta'^2}}\, \varphi', \qquad \frac{dp'_z}{dx_2} = \frac{e'_1 e_2}{x_1 x_2 M'_0}.
\end{aligned}
$$

The parameter β' is expected to be of order Λ_{QCD}.

Decay Constants

The decay constants for $J = 0, 1$ mesons are defined by the matrix elements

$$\begin{aligned}
\langle 0|A_\mu|P(P')\rangle &\equiv \mathcal{A}^P_\mu = i f_P P'_\mu, \\
\langle 0|V_\mu|S(P')\rangle &\equiv \mathcal{A}^S_\mu = f_S P'_\mu, \\
\langle 0|V_\mu|V(P',\varepsilon')\rangle &\equiv \mathcal{A}^V_\mu = M'_V f_V \varepsilon'_\mu, \\
\langle 0|A_\mu|\,^{3(1)}A(P',\varepsilon')\rangle &\equiv \mathcal{A}^{^3A(^1A)}_\mu = M'_{^3A(^1A)} f_{^3A(^1A)} \varepsilon'_\mu,
\end{aligned}$$

where the $^{2S+1}L_J = {}^1S_0$, $^3P_0$, $^3S_1$, $^3P_1$, $^1P_1$ and $^3P_2$ states of $q'_1\bar{q}_2$ mesons are denoted by P, S, V, ^3A, ^1A and T, respectively.

In the SU(N)-flavor limit ($m'_1 = m_2$) we should have vanishing f_S and $f_{^1A}$. It can be seen by applying equations of motion to the matrix element of the scalar resonance in to obtain

$$m_S^2 f_S = i(m'_1 - m_2)\langle 0|\bar{q}_1 q_2|S\rangle.$$

Note that a $^3P_2$ state cannot be produced by a current. It is based on the argument that the light $^3P_1$ and $^1P_1$ states transfer under charge conjugation as

$$M^b_a(^3P_1) \to M^a_b(^3P_1), \qquad M^b_a(^1P_1) \to -M^a_b(^1P_1), \quad (a = 1, 2, 3),$$

where the light axial-vector mesons are represented by a 3×3 matrix. Since the weak axial-vector current transfers as $(A_\mu)^b_a \to (A_\mu)^a_b$ under charge conjugation, it is clear that the decay constant of the $^1P_1$ meson vanishes in the SU(3) limit. This argument can be generalized to heavy axial-vector mesons. In fact, under similar charge conjugation argument $[(V_\mu)^b_a \to -(V_\mu)^a_b,\ M^b_a(^3P_0) \to M^a_b(^3P_0)]$ one can also prove the vanishing of f_S in the SU(N) limit.

In the heavy quark limit ($m_1' \to \infty$), the heavy quark spin s_Q decouples from the other degrees of freedom so that s_Q and the total angular momentum of the light antiquark j are separately good quantum numbers. It is more convenient to use the $L_J^j = P_2^{3/2}$, $P_1^{3/2}$, $P_1^{1/2}$ and $P_0^{1/2}$ basis. The first and the last of these states are ${}^3P_2$ and ${}^3P_0$, respectively, while

$$\left|P_1^{3/2}\right\rangle = \sqrt{\frac{2}{3}}\,\left|{}^1P_1\right\rangle + \frac{1}{\sqrt{3}}\,\left|{}^3P_1\right\rangle, \qquad \left|P_1^{1/2}\right\rangle = \frac{1}{\sqrt{3}}\,\left|{}^1P_1\right\rangle - \sqrt{\frac{2}{3}}\,\left|{}^3P_1\right\rangle.$$

Heavy quark symmetry (HQS) requires

$$f_V = f_P, \qquad f_{A^{1/2}} = f_S, \qquad f_{A^{3/2}} = 0,$$

where $P_1^{1/2}$ and $P_1^{3/2}$ states denoted by $A^{1/2}$ and $A^{3/2}$. These relations in the above equation can be understood from the fact that $(S_0^{1/2}, S_1^{1/2})$, $(P_0^{1/2}, P_1^{1/2})$ and $(P_1^{3/2}, P_2^{3/2})$ form three doublets in the HQ limit and that the tensor meson cannot be induced from the $V - A$ current.

Pseudoscalar:

To evaluate meson decay constants, the matrix element for the annihilation of a pseudoscalar state via axial currents can be written down and it has the expression

$$\mathcal{A}_\mu^P = -i^2 \frac{N_c}{(2\pi)^4} \int d^4 p_1' \frac{H_P'}{N_1' N_2} s_\mu^P,$$

where

$$\begin{aligned} s_\mu^P &\equiv \mathrm{Tr}[\gamma_\mu \gamma_5 (\not{p}_1' + m_1')\gamma_5(-\not{p}_2 + m_2)] \\ &= -4[m_1' P_\mu' + (m_2 - m_1')p_{1\mu}'], \end{aligned}$$

$N_1' = p_1'^2 - m_1'^2 + i\epsilon$ and $N_2 = p_2^2 - m_2^2 + i\epsilon$. Integrating out $p_1'^-$ in $\mathcal{A}_\mu^P$, If it is assumed that the vertex function H' has no pole in the upper complex $p_1'^-$ plane, the covariant calculation of meson properties and the calculation of the light-front formulism will give identical results at the one-loop level.

By closing the contour in the upper complex $p_1'^-$ plane and assuming that H_P' is analytic within the contour, the integration picks up a residue at $p_2 = \hat{p}_2$, where $\hat{p}_2^2 = m_2^2$. The other momentum is given by momentum conservation, $\hat{p}_1' = P' - \hat{p}_2$. Consequently, one has the following replacements:

$$
\begin{aligned}
N_1' &\to \hat{N}_1' = \hat{p}_1'^2 - m_1'^2 = x_1(M'^2 - M_0'^2), \\
H_M' &\to \hat{H}_M' = H_M'(\hat{p}_1'^2, \hat{p}_2^2) \equiv h_M', \\
W_M' &\to \hat{W}_M' = W_M'(\hat{p}_1'^2, \hat{p}_2^2) \equiv w_M', \\
\int \frac{d^4 p_1'}{N_1' N_2} H_M' s^M &\to -i\pi \int \frac{dx_2 d^2 p_\perp'}{x_2 \hat{N}_1'} h_M' \hat{s}^M,
\end{aligned}
$$

in a generic one-loop vacuum to particle M amplitude $\mathcal{A}_\mu^M$.

The matrix element $\mathcal{A}_\mu^P$ can be evaluated by using previous equations. However, $\mathcal{A}_\mu^P$ obtained in this way contains a spurious contribution proportional to $\tilde{\omega}^\mu = (\tilde{\omega}^-, \tilde{\omega}^+, \tilde{\omega}_\perp) = (2, 0, 0_\perp)$. It arises from the momentum decomposition of $\hat{p}_1'^\mu$

$$\begin{aligned}\hat{p}_1'^\mu &= (P' - \hat{p}_2)^\mu \\ &= x_1 P'^\mu + (0, 0, \vec{p}'_\perp)^\mu + \frac{1}{2}\left(x_2 P'^- - \frac{\vec{p}_{2\perp}^2 + m_2^2}{x_2 P'^+}\right)\tilde{\omega}^\mu.\end{aligned}$$

In fact, after the integration, p_1' can be expressed in terms of two external vectors, P' and $\tilde{\omega}$. Therefore, in the integrand of $\mathcal{A}_\mu^M$, one has

$$\begin{aligned}p'_{1\mu} &\doteq \frac{\tilde{\omega}\cdot p_1'}{\tilde{\omega}\cdot P'}P'_\mu + \frac{1}{\tilde{\omega}\cdot P'}\tilde{\omega}_\mu\left(P'\cdot p_1' - \frac{\tilde{\omega}\cdot p_1'}{\tilde{\omega}\cdot P'}P'^2\right) \\ &\doteq x_1 P'_\mu + \frac{1}{2\,\tilde{\omega}\cdot P'}\tilde{\omega}_\mu[-N_2 + N_1' + m_1'^2 - m_2^2 + (1-2x_1)M'^2].\end{aligned}$$

The symbol $\doteq$ in the above equation means that it is only true in the $\mathcal{A}^M$ integration.

There is one missing the contribution of the zero mode from the $p_1'^+ = 0$ region. The appearance of N_2 prompts an extra care in performing the $p_1'^-$ contour integration. This zero mode contribution provides a cue for the spurious term in $\mathcal{A}_\mu^M$. The inclusion of the zero mode contribution in $\mathcal{A}_\mu^M$ matrix elements amounts to the replacements

$$\hat{p}_1' \to x_1 P', \quad \hat{N}_2 \to \hat{N}_1' + m_1'^2 - m_2^2 + (1-2x_1)M'^2,$$

in the $\hat{S}^M$ under the integration.

$$f_P = \frac{N_c}{16\pi^3}\int dx_2 d^2p'_\perp \frac{h'_P}{x_1 x_2 (M'^2 - M_0'^2)} 4(m_1' x_2 + m_2 x_1).$$

f_P itself is free of zero mode contributions as its derivation does not involve the replacement of $\hat{N}_2$. With the explicit form of h'_P, the familiar expression of f_P in the conventional light-front approach

$$f_P = 2\frac{\sqrt{2N_c}}{16\pi^3}\int dx_2 d^2p'_\perp \frac{1}{\sqrt{x_1 x_2}\widetilde{M_0'}} (m_1' x_2 + m_2 x_1)\, \varphi'(x_2, p'_\perp),$$

is reproduced.

Scalar:

The decay constant of a scalar meson can be obtained in a similar manner. By using the corresponding Feynman rules,

$$\mathcal{A}^S_\mu = -i^2 \frac{N_c}{(2\pi)^4} \int d^4 p_1' \frac{H_S'}{N_1' N_2} \mathrm{Tr}[\gamma_\mu (\not{p}_1' + m_1')(-i)(-\not{p}_2 + m_2)].$$

Note that the trace ($\equiv s^S_\mu$) in the above equation is related to s^P_μ by the replacement of $m_2 \to -m_2$ and by adding an overall factor of $-i$. It follows that

$$f_S = \frac{N_c}{16\pi^3} \int dx_2 d^2 p'_\perp \frac{h_S'}{x_1 x_2 (M'^2 - M_0'^2)} 4(m_1' x_2 - m_2 x_1).$$

For $m_1' = m_2$, the meson wave function is symmetric with respect to x_1 and x_2, and hence $f_S = 0$, as it should be.

Vector:

The decay amplitude for a vector meson is given by

$$\mathcal{A}^V_\mu = -i^2 \frac{N_c}{(2\pi)^4} \int d^4p'_1 \frac{iH'_V}{N'_1 N_2} \mathrm{Tr}\left\{\gamma_\mu(\not{p}'_1 + m'_1)\left[\gamma_\nu - \frac{(p'_1 - p_2)_\nu}{W'_V}\right] \times(-\not{p}_2 + m_2)\right\}\varepsilon'^\nu.$$

Considering the case with the transverse polarization

$$\varepsilon(\pm) = \left(\frac{2}{P'^+}\varepsilon_\perp \cdot P'_\perp, 0, \varepsilon_\perp\right), \qquad \varepsilon_\perp = \mp\frac{1}{\sqrt{2}}(1, \pm i).$$

Contracting $\mathcal{A}^V_\mu$ with $\varepsilon^*(\pm)$, it leads to [1]

$$f_V = \frac{N_c}{4\pi^3 M'} \int dx_2 d^2p'_\perp \frac{h'_V}{x_1 x_2 (M'^2 - M'^2_0)} \times\left[x_1 M'^2_0 - m'_1(m'_1 - m_2) - p'^2_\perp + \frac{m'_1 + m_2}{w'_V} p'^2_\perp\right].$$

[1]When $\mathcal{A}^V_\mu$ is contracted with the longitudinal polarization vector $\varepsilon^\mu(0)$, f_V will receive additional contributions characterized by the B functions which give about 10% corrections to f_V for the vertex function h'_V.

Axial-vector:
$\mathcal{A}_\mu^{3A}$ ($\mathcal{A}_\mu^{1A}$) is related to $\mathcal{A}_\mu^{V}$ by a suitable replacement of $H'_V \to -H'_{^3A\,(^1A)}$ and $m_2 \to -m_2$, $W'_V \to -W'_{^3A\,(^1A)}$ in the trace (the $1/W'$ terms being kept in the ^1A case),

$$\begin{aligned} f_{^3A} &= -\frac{N_c}{4\pi^3 M'}\int dx_2 d^2p'_\perp \frac{h'_{^3A}}{x_1x_2(M'^2-M_0'^2)} \\ &\qquad \times\left[x_1M_0'^2 - m_1'(m_1'+m_2) - p_\perp'^2 - \frac{m_1'-m_2}{w'_{^3A}}\,p_\perp'^2\right], \\ f_{^1A} &= \frac{N_c}{4\pi^3 M'}\int dx_2 d^2p'_\perp \frac{h'_{^1A}}{x_1x_2(M'^2-M_0'^2)}\left(\frac{m_1'-m_2}{w'_{^1A}}\,p_\perp'^2\right). \end{aligned}$$

$f_{^1A}=0$ for $m_1'=m_2$. The SU(N)-flavor constraints on f_S and $f_{^1A}$ are satisfied.

In order to have a numerical values for decay constants, we need to specify the constituent quark masses and the parameter β appearing in the Gaussian-type wave function.

Form Factors

Form factors for **s-wave** to **s-wave** transitions:

Form factors for $P \to P, V$ transitions are defined by

$$\begin{aligned}
\langle P(P'')|V_\mu|P(P')\rangle &= P_\mu f_+(q^2) + q_\mu f_-(q^2), \\
\langle V(P'',\varepsilon'')|V_\mu|P(P')\rangle &= \epsilon_{\mu\nu\alpha\beta}\,\varepsilon''^{*\nu} P^\alpha q^\beta\, g(q^2), \\
\langle V(P'',\varepsilon'')|A_\mu|P(P')\rangle &= -i\left\{\varepsilon''^{*}_\mu f(q^2) + \varepsilon^{*\prime\prime}\cdot P\left[P_\mu a_+(q^2) + q_\mu a_-(q^2)\right]\right\},
\end{aligned}$$

where $P = P' + P''$, $q = P' - P''$ and $\epsilon_{0123} = 1$. These form factors are related to Bauer-Stech-Wirbel (BSW) form factors

$$\begin{aligned}
F_1^{PP}(q^2) &= f_+(q^2), \quad F_0^{PP}(q^2) = f_+(q^2) + \frac{q^2}{q\cdot P} f_-(q^2), \\
V^{PV}(q^2) &= -(M' + M'')\, g(q^2), \quad A_1^{PV}(q^2) = -\frac{f(q^2)}{M' + M''}, \\
A_2^{PV}(q^2) &= (M' + M'')\, a_+(q^2), \quad A_3^{PV}(q^2) - A_0^{PV}(q^2) = \frac{q^2}{2M''}\, a_-(q^2),
\end{aligned}$$

where the BSW form factors are defined by

$$\begin{aligned}
\langle P(P'')|V_\mu|P(P')\rangle &= \left(P_\mu - \frac{M'^2 - M''^2}{q^2} q_\mu\right) F_1^{PP}(q^2) \\
&\quad + \frac{M'^2 - M''^2}{q^2} q_\mu F_0^{PP}(q^2), \\
\langle V(P'', \varepsilon'')|V_\mu|P(P')\rangle &= -\frac{1}{M' + M''} \epsilon_{\mu\nu\alpha\beta} \varepsilon''^{*\nu} P^\alpha q^\beta V^{PV}(q^2), \\
\langle V(P'', \varepsilon'')|A_\mu|P(P')\rangle &= i\Big\{(M' + M'')\varepsilon''^*_\mu A_1^{PV}(q^2) - \frac{\varepsilon''^* \cdot P}{M' + M''} P_\mu A_2^{PV}(q^2) \\
&\quad - 2M'' \frac{\varepsilon''^* \cdot P}{q^2} q_\mu \left[A_3^{PV}(q^2) - A_0^{PV}(q^2)\right]\Big\},
\end{aligned}$$

with $F_1^{PP}(0) = F_0^{PP}(0)$, $A_3^{PV}(0) = A_0^{PV}(0)$, and

$$A_3^{PV}(q^2) = \frac{M' + M''}{2M''} A_1^{PV}(q^2) - \frac{M' - M''}{2M''} A_2^{PV}(q^2).$$

This parametrization has the advantage that the q^2 dependence of the form factors is governed by the resonances of the same spin.

At $q^2 = 0$, the form factor $f_+(0)$ is given by

$$f_+(q^2) = \frac{N_c}{16\pi^3}\int dx_2 d^2p'_\perp \frac{h'_P h''_P}{x_2 \hat{N}'_1 \hat{N}''_1}\Big[x_1(M'^2_0 + M''^2_0) + x_2 q^2 - x_2(m'_1 - m''_1)^2 - x_1(m'_1 - m_2)^2 - x_1(m''_1 - m_2)^2\Big].$$

$$f_-(q^2) = \frac{N_c}{16\pi^3}\int dx_2 d^2p'_\perp \frac{2h'_P h''_P}{x_2 \hat{N}'_1 \hat{N}''_1}\Big\{ - x_1 x_2 M'^2 - p'^2_\perp - m'_1 m_2 + (m''_1 - m_2)(x_2 m'_1 + x_1 m_2) + 2\frac{q\cdot P}{q^2}\left(p'^2_\perp + 2\frac{(p'_\perp \cdot q_\perp)^2}{q^2}\right) + 2\frac{(p'_\perp \cdot q_\perp)^2}{q^2} - \frac{p'_\perp \cdot q_\perp}{q^2}\Big[M''^2 - x_2(q^2 + q\cdot P) - (x_2 - x_1)M'^2 + 2x_1 M'^2_0 - 2(m'_1 - m_2)(m'_1 + m''_1)\Big]\Big\}.$$

$P \to V$ transition form factors given by

$$
\begin{aligned}
g(q^2) &= -\frac{N_c}{16\pi^3}\int dx_2 d^2p'_\perp \frac{2h'_P h''_V}{x_2 \hat{N}'_1 \hat{N}''_1}\Bigg\{x_2 m'_1 + x_1 m_2 + (m'_1 - m''_1)\frac{p'_\perp \cdot q_\perp}{q^2} \\
&\quad + \frac{2}{w''_V}\left[p'^2_\perp + \frac{(p'_\perp \cdot q_\perp)^2}{q^2}\right]\Bigg\}, \\
f(q^2) &= \frac{N_c}{16\pi^3}\int dx_2 d^2p'_\perp \frac{h'_P h''_V}{x_2 \hat{N}'_1 \hat{N}''_1}\Bigg\{2x_1(m_2 - m'_1)(M'^2_0 + M''^2_0) \\
&\quad -4x_1 m''_1 M'^2_0 + 2x_2 m'_1 q\cdot P + 2m_2 q^2 - 2x_1 m_2 (M'^2 + M''^2) \\
&\quad +2(m'_1 - m_2)(m'_1 + m''_1)^2 + 8(m'_1 - m_2)\left[p'^2_\perp + \frac{(p'_\perp \cdot q_\perp)^2}{q^2}\right] \\
&\quad +2(m'_1 + m''_1)(q^2 + q\cdot P)\frac{p'_\perp \cdot q_\perp}{q^2} - 4\frac{q^2 p'^2_\perp + (p'_\perp \cdot q_\perp)^2}{q^2 w''_V} \\
&\quad \Bigg[2x_1(M'^2 + M'^2_0) - q^2 - q\cdot P - 2(q^2 + q\cdot P)\frac{p'_\perp \cdot q_\perp}{q^2} \\
&\quad -2(m'_1 - m''_1)(m'_1 - m_2)\Bigg]\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
a_+(q^2) &= \frac{N_c}{16\pi^3}\int dx_2 d^2p'_\perp \frac{2h'_P h''_V}{x_2\hat{N}'_1\hat{N}''_1}\Bigg\{(x_1-x_2)(x_2m'_1+x_1m_2)-[2x_1m_2 \\
&\quad +m''_1+(x_2-x_1)m'_1]\frac{p'_\perp\cdot q_\perp}{q^2}-2\frac{x_2q^2+p'_\perp\cdot q_\perp}{x_2q^2w''_V}\Big[p'_\perp\cdot p''_\perp \\
&\quad +(x_1m_2+x_2m'_1)(x_1m_2-x_2m''_1)\Big]\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
a_-(q^2) &= \frac{N_c}{16\pi^3}\int dx_2 d^2p'_\perp \frac{h'_P h''_V}{x_2 \hat{N}'_1 \hat{N}''_1}\Bigg\{2(2x_1-3)(x_2 m'_1 + x_1 m_2) \\
&\quad -8(m'_1 - m_2)\left[\frac{p'^2_\perp}{q^2} + 2\frac{(p'_\perp\cdot q_\perp)^2}{q^4}\right] - [(14-12x_1)m'_1 - 2m''_1 \\
&\quad -(8-12x_1)m_2]\frac{p'_\perp\cdot q_\perp}{q^2} + \frac{4}{w''_V}\Bigg([M'^2 + M''^2 - q^2 + 2(m'_1 - m_2) \\
&\quad \times(m''_1 + m_2)](A_3^{(2)} + A_4^{(2)} - A_2^{(1)}) + Z_2(3A_2^{(1)} - 2A_4^{(2)} - 1) \\
&\quad +\frac{1}{2}[x_1(q^2 + q\cdot P) - 2M'^2 - 2p'_\perp\cdot q_\perp - 2m'_1(m''_1 + m_2) \\
&\quad -2m_2(m'_1 - m_2)](A_1^{(1)} + A_2^{(1)} - 1) + q\cdot P\left[\frac{p'^2_\perp}{q^2} + \frac{(p'_\perp\cdot q_\perp)^2}{q^4}\right] \\
&\quad (4A_2^{(1)} - 3)\Bigg)\Bigg\},
\end{aligned}
$$

Form factors for **s-wave** to **p-wave** transitions:

The general expressions for P to low-lying p-wave meson transitions

$$
\begin{aligned}
\langle S(P'')|A_\mu|P(P')\rangle &= i\Big[u_+(q^2)P_\mu + u_-(q^2)q_\mu\Big],\\
\langle A^{1/2}(P'',\varepsilon'')|V_\mu|P(P')\rangle &= i\Big\{\ell_{1/2}(q^2)\varepsilon''^*_\mu + \varepsilon''^*\cdot P[P_\mu c_+^{1/2}(q^2)\\
&\quad +q_\mu c_-^{1/2}(q^2)]\Big\},\\
\langle A^{1/2}(P'',\varepsilon'')|A_\mu|P(P')\rangle &= -q_{1/2}(q^2)\epsilon_{\mu\nu\alpha\beta}\varepsilon''^{*\nu}P^\alpha q^\beta,\\
\langle A^{3/2}(P'',\varepsilon'')|V_\mu|P(P')\rangle &= i\Big\{\ell_{3/2}(q^2)\varepsilon''^*_\mu + \varepsilon''^*\cdot P[P_\mu c_+^{3/2}(q^2)\\
&\quad +q_\mu c_-^{3/2}(q^2)]\Big\},\\
\langle A^{3/2}(P'',\varepsilon'')|A_\mu|P(P')\rangle &= -q_{3/2}(q^2)\epsilon_{\mu\nu\alpha\beta}\varepsilon''^{*\nu}P^\alpha q^\beta.
\end{aligned}
$$

$\ell_{1/2(3/2)}, c_+^{1/2(3/2)}, c_-^{1/2(3/2)}$ and $q_{1/2(3/2)}$ are defined forthe heavy $P_1^{1/2}$ ($P_1^{3/2}$) state. For light axial-vector mesons: $L-S$ coupled states $^1P_1$ and $^3P_1$ denoted by the particles ^1A and ^3A.
$u_+(q^2), u_-(q^2)$ and $k(q^2)$ are dimensionless.

Defining dimensionless form factors by

$$
\begin{aligned}
\langle S(P'')|A_\mu|P(P')\rangle &= -i\left[\left(P_\mu - \frac{M'^2 - M''^2}{q^2}\,q_\mu\right)F_1^{PS}(q^2) \right.\\
&\left. + \frac{M'^2 - M''^2}{q^2}\,q_\mu\,F_0^{PS}(q^2)\right],\\
\langle A(P'',\varepsilon'')|V_\mu|P(P')\rangle &= -i\left\{(m_P - m_A)\varepsilon^*_\mu V_1^{PA}(q^2) - \frac{\varepsilon^*\cdot P'}{m_P - m_A}\right.\\
&\times P_\mu V_2^{PA}(q^2) - 2m_A\frac{\varepsilon^*\cdot P'}{q^2}q_\mu\left[V_3^{PA}(q^2)\right.\\
&\left.\left. - V_0^{PA}(q^2)\right]\right\},\\
\langle A(P'',\varepsilon'')|A_\mu|P(P')\rangle &= -\frac{1}{m_P - m_A}\,\epsilon_{\mu\nu\rho\sigma}\varepsilon^{*\nu}P^\rho q^\sigma A^{PA}(q^2),
\end{aligned}
$$

with

$$
V_3^{PA}(q^2) = \frac{m_P - m_A}{2m_A}\,V_1^{PA}(q^2) - \frac{m_P + m_A}{2m_A}\,V_2^{PA}(q^2), \quad V_3^{PA}(0) = V_0^{PA}(0).
$$

$$
\begin{aligned}
F_1^{PS}(q^2) &= -u_+(q^2), \quad F_0^{PS}(q^2) = -u_+(q^2) - \frac{q^2}{q\cdot P} u_-(q^2), \\
A^{PA}(q^2) &= -(M' - M'')\, q(q^2), \quad V_1^{PA}(q^2) = -\frac{\ell(q^2)}{M' - M''}, \\
V_2^{PA}(q^2) &= (M' - M'')\, c_+(q^2), \quad V_3^{PA}(q^2) - V_0^{PA}(q^2) = \frac{q^2}{2M''}\, c_-(q^2).
\end{aligned}
$$

In above equations, the axial-vector meson A stands for $A^{1/2}$ or $A^{3/2}$.

The $P \to S\,(A)$ transition form factors can obtained by suitable modifications on $P \to P\,(V)$. The $P \to S$ transition form factors are related to $f_\pm$ by

$$u_\pm = -f_\pm(m_1'' \to -m_1'', h_P'' \to h_S'').$$

these form factors can be obtained from $P \to P$ by the replacements,

$$\begin{aligned}
u_+(q^2) &= \frac{N_c}{16\pi^3}\int dx_2 d^2p'_\perp \frac{h'_P h''_S}{x_2\hat{N}'_1\hat{N}''_1}\Big[-x_1(M_0'^2 + M_0''^2) - x_2 q^2 \\
&\quad + x_2(m_1' + m_1'')^2 + x_1(m_1' - m_2)^2 + x_1(m_1'' + m_2)^2\Big], \\
u_-(q^2) &= \frac{N_c}{16\pi^3}\int dx_2 d^2p'_\perp \frac{2h'_P h''_S}{x_2\hat{N}'_1\hat{N}''_1}\Bigg\{x_1x_2M'^2 + p_\perp'^2 + m_1'm_2 + (m_1'' + m_2) \\
&\quad \times(x_2m_1' + x_1m_2) - 2\frac{q\cdot P}{q^2}\left(p_\perp'^2 + 2\frac{(p'_\perp\cdot q_\perp)^2}{q^2}\right) - 2\frac{(p'_\perp\cdot q_\perp)^2}{q^2} \\
&\quad + \frac{p'_\perp\cdot q_\perp}{q^2}\Big[M''^2 - x_2(q^2 + q\cdot P) - (x_2 - x_1)M'^2 + 2x_1M_0'^2 \\
&\quad - 2(m_1' - m_2)(m_1' - m_1'')\Big]\Bigg\}.
\end{aligned}$$

Similarly, the analytic expressions for $P \to A$ transition form factors can be obtained from that of $P \to V$ ones by the following replacements:

$$
\begin{aligned}
\ell^{^3A,^1A}(q^2) &= f(q^2) \text{ with } (m_1'' \to -m_1'',\, h_V'' \to h''_{^3A,^1A},\, w_V'' \to w''_{^3A,^1A}),\\
q^{^3A,^1A}(q^2) &= g(q^2) \text{ with } (m_1'' \to -m_1'',\, h_V'' \to h''_{^3A,^1A},\, w_V'' \to w''_{^3A,^1A}),\\
c_+^{^3A,^1A}(q^2) &= a_+(q^2) \text{ with } (m_1'' \to -m_1'',\, h_V'' \to h''_{^3A,^1A},\, w_V'' \to w''_{^3A,^1A}),\\
c_-^{^3A,^1A}(q^2) &= a_-(q^2) \text{ with } (m_1'' \to -m_1'',\, h_V'' \to h''_{^3A,^1A},\, w_V'' \to w''_{^3A,^1A}).
\end{aligned}
$$

The replacement of $m_1'' \to -m_1''$ should not be applied to m_1'' in w'' and h''. These form factors can be expressed in the $P_1^{3/2}$ and $P_1^{1/2}$ basis.

- Because of the condition $q^+ = 0$, form factors are known only for spacelike momentum transfer $q^2 = -q_\perp^2 \leqslant 0$, whereas only the timelike form factors are relevant for the physical decay processes.
- The form factors as explicit functions of q^2 in the spacelike region and then analytically continue them to the timelike region.
- Another approach is to construct a double spectral representation for form factors at $q^2 < 0$ and then analytically continue it to $q^2 > 0$ region .
- Within a specific model, form factors obtained directly from the timelike region (with $q^+ > 0$) are identical to the ones obtained by the analytic continuation from the spacelike region.
- In principle, form factors at $q^2 > 0$ can be evaluated directly in the frame where the momentum transfer is purely longitudinal, i.e., $q_\perp = 0$, so that $q^2 = q^+ q^-$ covers the entire range of momentum transfer.
- The price one has to pay is that, besides the conventional valence-quark contribution, one must also consider the non-valence configuration (or the so-called Z-graph) arising from quark-pair creation from the vacuum.

- However, a reliable way of estimating the Z-graph contribution is still lacking unless one works in a specific model. Fortunately, this additional non-valence contribution vanishes in the frame where the momentum transfer is purely transverse i.e., $q^+ = 0$.
- The momentum dependence of form factors in the spacelike region can be well parameterized and reproduced in the following three-parameter form:

$$F(q^2) = \frac{F(0)}{1 - a(q^2/m^2_{B(D)}) + b(q^2/m^2_{B(D)})^2},$$

  for $P \to M$ transitions, where F stands for the relevant form factors appearing in these transitions.
  The parameters a, b and $F(0)$ are first determined in the spacelike region, then employed this parametrization to determine the physical form factors at $q^2 \geqslant 0$. In practice, the parameters a, b and $F(0)$ are obtained by performing a 5-parameter fit to the form factors in the range $-20\,\text{GeV}^2 \leqslant q^2 \leqslant 0$ for B decays and $-10\,\text{GeV}^2 \leqslant q^2 \leqslant 0$ for D decays.

The End