Transformative A_4 Mixing of Neutrinos with CP Violation

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1. Introduction and Motivation

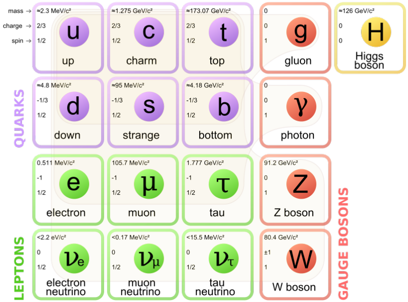


Figure: Standard Model

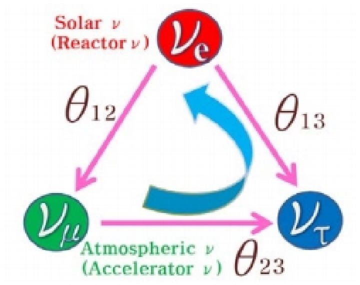


Figure: Neutrino Oscillation

- Neutrinos should be massive!
- We need an appropriate theory to fill the gap.

} We need BSM!

1. Introduction and Motivation

Quantity	Three-neutrino mixing parameters from pdg
$\Delta m^2_{sun} = \Delta m^2_{21}(10^{-5} eV^2)$	7.53 ± 0.18
$\Delta m^2_{atm} = \Delta m^2_{32}(10^{-3} eV^2)$	2.42 ± 0.06
$\sin^2 \theta_{12}$	0.304 ± 0.014
$\sin^2 2\theta_{12}$	0.846 ± 0.021
$\sin^2 \theta_{23}$	$0.514^{+0.055}_{-0.056}$
$\sin^2 2\theta_{23}$	$0.999^{+0.001}_{-0.018}$
$\sin^2 \theta_{13}$	0.0219 ± 0.0012
$\sin^2 2\theta_{13}$	0.085 ± 0.005
δ_{CP}	$\pm\pi/2$

Table: Neutrino oscillation data

Oscillation experiments do NOT provide information about

- absolute neutrino mass scale
- Dirac/Majorana nature of neutrinos

1. Introduction and Motivation

We don't know absolute ν mass scale. So we have two possible scenarios.

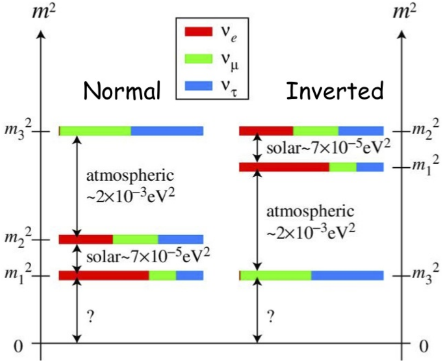


Figure: Normal and Inverted hierarchy

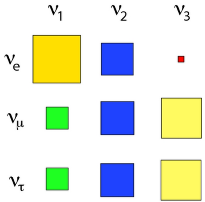
From this graph, we can naturally come up with neutrino mixing.

1. Introduction and Motivation

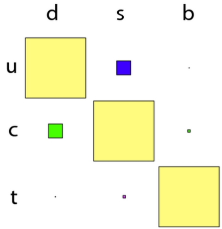
Neutrino mixing is important because it could provide new clues for the understanding of the flavor problem.

Neutrino mixing pattern is completely different that of quark mixing.

Neutrino Mixing



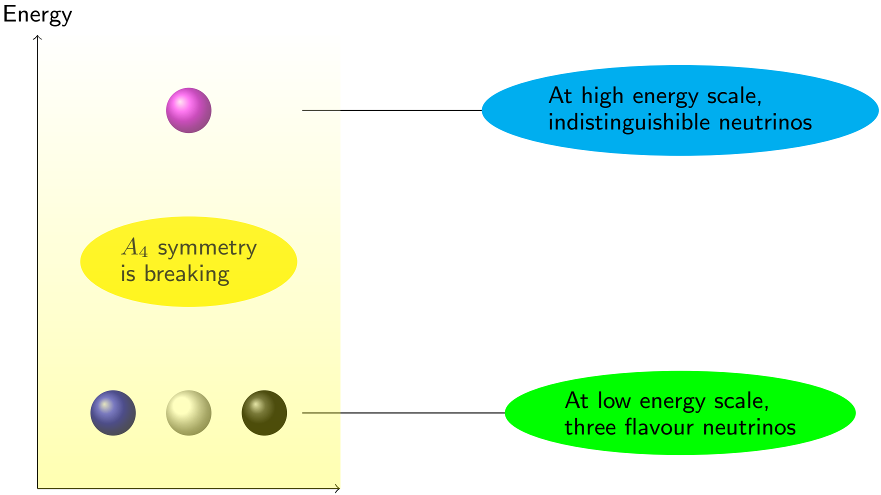
Quark Mixing



From neutrino mixing, we could expect a specific symmetry!

1. Introduction and Motivation

Then you can ask me why you take a specific symmetry as for neutrino research.



1. Introduction and Motivation

In order to read a specific symmetry from neutrino mixing, lots of special groups have been studied by neutrino theorists.

Group	d	Irr. Repr.'s	Presentation
$D_3 \sim S_3$	6	$1, 1', 2$	$A^3 = B^2 = (AB)^2 = 1$
D_4	8	$1_1, \cdot, 1_4, 2$	$A^4 = B^2 = (AB)^2 = 1$
D_7	14	$1, 1', 2, 2', 2''$	$A^7 = B^2 = (AB)^3 = 1$
A_4	12	$1, 1', 1'', 3$	$A^3 = B^2 = (AB)^3 = 1$
$A_5 \sim PSL_2(5)$	60	$1, 3, 3', 4, 5$	$A^3 = B^2 = (BA)^5 = 1$
T'	24	$1, 1', 1'', 2, 2', 2'', 3$	$A^3 = (AB)^3 = R^2 = 1, B^2 = R$
S_4	24	$1, 1', 2, 3, 3'$	$BM \ : \ A^4 = B^2 = (AB)^3 = 1$ $TB \ : \ A^3 = B^4 = (BA^2)^2 = 1$
$\Delta(27) \sim Z_3 \rtimes Z_3$	27	$1_1, \cdot, 1_9, 3, \bar{3}$	
$PSL_2(7)$	168	$1, 3, \bar{3}, 6, 7, 8$	$A^3 = B^2 = (BA)^7 = (B^{-1}A^{-1}BA)^4 = 1$
$T_7 \sim Z_7 \rtimes Z_3$	21	$1, 1', \bar{1}', 3, \bar{3}$	$A^7 = B^3 = 1, AB = BA^4$

Table: Some small discrete groups used for model building.

- Tri-Bimaximal mixing : mixing equally ν_e with ν_μ, and ν_τ
- Bimaximal mixing : mixing equally ν_μ with ν_τ

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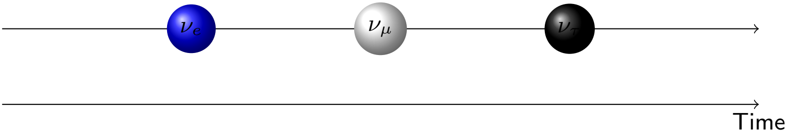
2. Special Patterns of Neutrino Mixing



To talk about neutrino mixing, let me define flavour and mass eigenstates.

- $\nu_f = (\nu_e,\ \nu_\mu,\ \nu_\tau)$
- $\nu_m = (\nu_1,\ \nu_2,\ \nu_3)$

Then, we can say relation between flavour and mass eigenstates like below

$$\nu_f = U_{PMNS}\nu_m$$

So PMNS mixing matrix is given like below

$$U_{PMNS} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{CP}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{CP}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- $c_{ij} = \cos\theta_{ij},\ s_{ij} = \sin\theta_{ij}$

2. Special Patterns of Neutrino Mixing

The general form of the neutrino mass matrix with U is given by:

$$\mathcal{M}_\nu^{(e,\mu,\tau)} = U^* \mathcal{M}_\nu^{(1,2,3)} U^\dagger$$

Let me consider a special form of the Majorana neutrino mass matrix which first appeared in 2002[1][2]

$$\mathcal{M}_\nu^{(e,\mu,\tau)} = \begin{pmatrix} A & C & C^* \\ C & D^* & B \\ C^* & B & D \end{pmatrix}$$

where A, B are real.

This allows three important properties.

- $\theta_{13} \neq 0$
- $\theta_{23} = \pi/4$
- $\delta_{CP} = \pm\pi/2$

This pattern was shown[3] to be protected by a symmetry, i.e.

- $e \rightarrow e$
- $\mu \leftrightarrow \tau$ with CP conjugation

[1] K. S. Babu, E. Ma, and J. W. F. Valle, Phys. Lett. **B552**, 207 (2003)
[2] E. Ma, Phys. Rev. **D66**, 117301 (2002)
[3] W. Grimus and L. Lavoura, Phys. Lett. **B579**, 113 (2004)

2. Special Patterns of Neutrino Mixing

This pattern was shown to be protected by a symmetry, i.e.

- $e \;\rightarrow\; e$
- $\mu \;\leftrightarrow\; \tau$ with CP conjugation

explaining roughly the above symmetry,

$$\mathcal{M}_\nu^{(e,\mu,\tau)} = \begin{pmatrix} \mathcal{M}_{ee} & \mathcal{M}_{e\mu} & \mathcal{M}_{e\tau}^* \\ \mathcal{M}_{\mu e} & \mathcal{M}_{\mu\mu}^* & \mathcal{M}_{\mu\tau} \\ \mathcal{M}_{\tau e}^* & \mathcal{M}_{\tau\mu} & \mathcal{M}_{\tau\tau} \end{pmatrix}$$

$$\downarrow (e \rightarrow e, \mu \leftrightarrow \tau)$$

$$\mathcal{M}_\nu^{(e,\mu,\tau)} = \begin{pmatrix} \mathcal{M}_{ee} & \mathcal{M}_{e\tau} & \mathcal{M}_{e\mu}^* \\ \mathcal{M}_{\tau e} & \mathcal{M}_{\tau\tau}^* & \mathcal{M}_{\tau\mu} \\ \mathcal{M}_{\mu e}^* & \mathcal{M}_{\mu\tau} & \mathcal{M}_{\mu\mu} \end{pmatrix}$$

With the knowledge that $\theta_{13} \neq 0$, this extended symmetry is now the subject of many studies, which began with generalized S_4[4]

In this paper, I show how $\theta_{23} = \pi/4$ and $\delta_{CP} = \pm\pi/2$ may be obtained in a very general way, using the familiar unitary 3×3 transformation

$$U_\omega = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$$

where $\omega = \exp(2\pi i/3) = -1/2 + i\sqrt{3}/2$, which is derivable from A_4 as shown in Ref[5]

[4] R. N. Mohapatra and C. C. Nishi, Phys. Rev. **D86**, 073007 (2012)

[5] E. Ma and G. Rajasekaran, Phys. Rev. **D64**, 113012 (2001)

2. Special Patterns of Neutrino Mixing

The idea is very simple.

Consider the product of $U_\omega \mathcal{O}$, where $\mathcal{O}$ is a real orthogonal 3×3 matrix, i.e.

$$\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} o_{11} & o_{12} & o_{13} \\ o_{21} & o_{22} & o_{23} \\ o_{31} & o_{32} & o_{33} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} = U$$

Calculating l.h.s.,

$$U = \frac{1}{\sqrt{3}} \begin{pmatrix} o_{11} + o_{21} + o_{31} & o_{12} + o_{22} + o_{32} & o_{13} + o_{23} + o_{33} \\ o_{11} + \omega o_{21} + \omega^2 o_{31} & o_{12} + \omega o_{22} + \omega^2 o_{32} & o_{13} + \omega o_{23} + \omega^2 o_{33} \\ o_{11} + \omega^2 o_{21} + \omega o_{31} & o_{12} + \omega^2 o_{22} + \omega o_{32} & o_{13} + \omega^2 o_{23} + \omega o_{33} \end{pmatrix}$$

It is clear that $u_{2i}^* = u_{3i}$ for $i = 1, 2, 3$. Comparing this with the PDG convention of the neutrino mixing matrix, i.e.

$$U_{\mathsf{SF}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13}s_{13}e^{-i\delta} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

2. Special Patterns of Neutrino Mixing

$$U_{\mathsf{SF}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13}s_{13}e^{-i\delta} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

where

- $s_{ij} = \sin\theta_{ij}$, $c_{ij} = \cos\theta_{ij}$
- θ_{ij} is the mixing angle which mixes mass eigenstates ν_i and ν_j

Then, we can write $U = PU_{\mathsf{SF}}P'$, where P and P' are diagonal phase matrices.

The restrictions $U_{2i} = U^*_{3i}$ for $i = 1, 2, 3$ lead to the constraints

$$|(U_{\mathsf{SF}})_{2i}| = |(U_{\mathsf{SF}})_{3i}|$$

for $i = 1, 2, 3$, which are expressed by

$$|-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta}| = |s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta}|$$

$$|c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta}| = |-c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta}|$$

$$|s_{23}c_{13}| = |c_{23}c_{13}|$$

By solving these equations, we find

$$c_{23}^2 = s_{23}^2, \quad \cos\delta = 0$$

2. Special Patterns of Neutrino Mixing

$$c_{23}^2 = s_{23}^2, \quad \cos\delta = 0$$

or

$$\theta_{23} = \pi/4 \ , \ \delta_{CP} = \pm\pi/2$$

This was first pointed out in Refs[6][7]

To obtain this result, the necessary condition is that the 3×3 Majorana neutrino mass matrix $\mathcal{M}_\nu$ must be diagonalized by an orthogonal matrix in the A_4 basis.

Obviously it will be so if $\mathcal{M}_\nu$ is purely real. In that case, in the (e, μ, τ) basis, it is given by

$$\mathcal{M}_\nu^{(e,\mu,\tau)} = U_\omega \begin{pmatrix} a & c & e \\ c & d & b \\ e & b & f \end{pmatrix} U_\omega^T = \begin{pmatrix} A & C & C^* \\ C & D^* & B \\ C^* & B & D \end{pmatrix}$$

where

- $A \ = \ (a + 2b + 2c + d + 2e + f)/3,$
- $B \ = \ (a - b - c + d - e + f)/3,$
- $C \ = \ (a - b - \omega^2 c + \omega d - \omega e + \omega^2 f)/3,$
- $D \ = \ (a + 2b + 2\omega^2 c + \omega d + 2\omega e + \omega^2 f)/3.$

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[6] K. Fukuura, T. Miura, E. Takasugi, and M. Yoshimura, Phys. Rev. **D61**, 073002 (2000)

[7] T. Miura, E. Takasugi, and M. Yoshimura, Phys. Rev. **D63**, 013001 (2001)

2. Special Patterns of Neutrino Mixing

I maybe guess you will have a hard time to make sense what I wrote down below.

- $A = (a + 2b + 2c + d + 2e + f)/3$,
- $B = (a - b - c + d - e + f)/3$,
- $C = (a - b - \omega^2 c + \omega d - \omega e + \omega^2 f)/3$,
- $D = (a + 2b + 2\omega^2 c + \omega d + 2\omega e + \omega^2 f)/3$.

Let's see these equations in matrix form.

$$\mathcal{M}_\nu^{(e,\mu,\tau)} = a\frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + b\frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} + c\frac{1}{3}\begin{pmatrix} 2 & -\omega^2 & -\omega \\ -\omega^2 & 2\omega & -1 \\ -\omega & -1 & 2\omega^2 \end{pmatrix}$$
$$d\frac{1}{3}\begin{pmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{pmatrix} + e\frac{1}{3}\begin{pmatrix} 2 & -\omega & -\omega^2 \\ -\omega & 2\omega^2 & -1 \\ -\omega^2 & -1 & 2\omega \end{pmatrix} + f\frac{1}{3}\begin{pmatrix} 1 & \omega^2 & \omega \\ \omega^2 & \omega & 1 \\ \omega & 1 & \omega^2 \end{pmatrix}$$

You will be maybe more confusing. But please be more patient for a short time.

2. Special Patterns of Neutrino Mixing

In the context of A_4, efforts prior to 2011 were concentrated on how to achieve $c = e = 0$ and $d = f$ for tribimaximal mixing without a necessarily real $\mathcal{M}_\nu$,

$$\mathcal{M}_\nu^{(e,\mu,\tau)} = \frac{1}{3}\begin{pmatrix} a+2b+2d & a-b-d & a-b-d \\ a-b-d & a+2b-d & a-b+2d \\ a-b-d & a-b+2d & a+2b-d \end{pmatrix} \Longrightarrow$$

μ-τ symmetry or 2-3 symmetry

i.e. a residual Z_2 symmetry in the neutrino sector which coexists with the residual Z_3 symmetry implied by U_ω in the charged lepton sector.

$$\mathcal{M}_\nu^{(e,\mu,\tau)} = \underbrace{\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}}_{Z_3} \begin{pmatrix} a & 0 & 0 \\ 0 & d & b \\ 0 & b & d \end{pmatrix} \underbrace{\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}}_{Z_3}$$

in other words,

$$\begin{pmatrix} e \\ \nu_e \end{pmatrix} \quad \begin{pmatrix} \mu \\ \nu_\mu \end{pmatrix} \quad \begin{pmatrix} \tau \\ \nu_\tau \end{pmatrix}$$

$\nu_\mu \leftrightarrow \nu_\tau$, Z_2 by maximal mixing
but $m_\mu \neq m_\tau$, we need Z_3 in the charged-lepton sector, and A_4 might be bridge to connect them.

2. Special Patterns of Neutrino Mixing

This clash or misalignment of residual symmetries is the origin of a basic theoretical problem which has no simple solution.

In hindsight, it is a powerful argument against the naive expectation of an exact tribimaximal form of the neutrino mixing matrix

Here A_4 serves simply as a link for a (real) neutrino mass matrix without any symmetry to the charged-lepton sector.

i.e. U_ω leads to two verifiable specific predictions, i.e.

- $\theta_{23} = \pi/4$
- $\delta_{CP} = \pm\pi/2$

which agree well with present data.

In Ref[8], $c = e = 0$ is again assumed but d and f are not set equal. In this way $\theta_{13} \neq 0$ is obtained and the further assumption that a, b, d, f are real leads to $\theta_{23} = \pi/4$ and $\delta_{CP} = \pm\pi/2$

[8] X.-G. He, arXiv:1504.01560 [hep-ph]

2. Special Patterns of Neutrino Mixing

Here I answer the new important question of how an arbitrary complex neutrino mass matrix can be guaranteed to be purely real, without imposing explicit CP conservation.

The key is of course the origin of $\mathcal{O}$ which obiviously would be the result of diagonalizing a real 3×3 mass matrix. The only guaranteed such mass matrix is that of three real scalars.

Hence the quest for $\mathcal{O}$ leads inexorably to a mechanism by which neutrino masses come from three real scalars.

This is the significance of equation($U_\omega \mathcal{O} = U$).

In the following, I will show that it may be achieved naturally together with the appearance of U_ω in a radiative implementation[9][10] of neutrino and charged-lepton masses through dark matter (scotogenic), using $only$ the one Higgs doublet of the standard model (SM).

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[9] E. Ma, Phys. Rev. **D73**, 077301 (2006)

[10] E. Ma, Phys. Rev. Lett **112**, 091801 (2014)

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3. The A_4 symmetry

The A_4 symmetry is group of the even permutations of S_4.

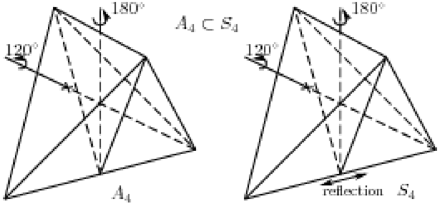

Figure: The A_4 symmetry

The A_4 group is the smallest non-Abelian group. And the order of this group is $(4!)/2 = 12$.

Here, we present A_4 group, which consists of all even permutations among four objects, (x_1, x_2, x_3, x_4),

$$(x_1, x_2, x_3, x_4) \;\rightarrow\; (x_i, x_j, x_k, x_l)$$

$$\begin{array}{lcll}
a_1 & : & (1,2,3,4) & \\
a_2 & : & (2,3,1,4) & (1 \to 2, 2 \to 3) \\
a_3 & : & (2,4,3,1) & (1 \to 2, 2 \to 4) \\
a_4 & : & (3,2,4,1) & (1 \to 3, 3 \to 4) \\
b_1 & : & (3,1,2,4) & (1 \to 3, 3 \to 2) \\
b_2 & : & (4,1,3,2) & (1 \to 4, 4 \to 2) \\
b_3 & : & (4,2,1,3) & (1 \to 4, 4 \to 3) \\
b_4 & : & (1,3,4,2) & (2 \to 3, 3 \to 4) \\
c_1 & : & (1,4,2,3) & (2 \to 4, 4 \to 3) \\
c_2 & : & (2,1,4,3) & (1 \to 2), (3 \to 4) \\
c_3 & : & (3,4,1,2) & (1 \to 3), (2 \to 4) \\
c_4 & : & (4,3,2,1) & (1 \to 4), (2 \to 3)
\end{array}$$

3. The A_4 symmetry

It is obvious that $x_1 + x_2 + x_3 + x_4$ is invariant under any permutation of A_4, that is, a trivial singlet. Thus we use the vector space, which is orthogonal to this singlet direction,

$$\mathbf{3} \ : \ \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} x_1 + x_2 - x_3 - x_4 \\ x_1 - x_2 + x_3 - x_4 \\ x_1 - x_2 - x_3 + x_4 \end{pmatrix},$$

in order to construct matrix representations of A_4, that is, a triplet representation. In this triplet vector space, all of A_4 elements are represented by the following matrices.

$$a_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ a_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \ a_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \ a_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$b_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \ b_2 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \ b_3 = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \ b_4 = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$c_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \ c_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix} \ c_3 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \ c_4 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$$

3. The A_4 symmetry

They are classified by the conjugacy classes as

$$\begin{aligned} C_1 &: \quad a_1, \qquad h=1, \\ C_3 &: \quad a_2, a_3, a_4, \quad h=2, \\ C_4 &: \quad b_1, b_2, b_3, b_4, \quad h=3, \\ C_{4'} &: \quad c_1, c_2, c_3, c_4, \quad h=3, \end{aligned}$$

There are four conjugacy classes and there must be four irreducible representations, i.e.

$$m_1 + m_2 + m_3 + \cdots = 4$$

. The orthogonality relation requires

$$\sum_\alpha [\chi_\alpha(C_1)]^2 = \sum_n m_n n^2 = m_1 + 4m_2 + \cdots = 12$$

,

for m_i, which satisfy $m_1 + m_2 + m_3 + \cdots = 4$.

Then, we have obtain a solution, $(m_1, m_2, m_3) = (3, 0, 1)$.

That is, the A_4 group has three singlets, $\mathbf{1}, \mathbf{1}'$, and $\mathbf{1}''$, and a single $\mathbf{3}$.

Another algebraic definition of A_4 is often used in the literature. We denote

- $a_1 = e$
- $a_2 = s$
- $b_1 = t$.

They satisfy the following algebraic relations,

$$s^2 = t^3 = (st)^3 = e$$

Before to go on

One assigns **leptons** to the **four inequivalent representations of A_4**:

$$\underbrace{\begin{pmatrix} e \\ \nu_e \end{pmatrix} \quad \begin{pmatrix} \mu \\ \nu_\mu \end{pmatrix} \quad \begin{pmatrix} \tau \\ \nu_\tau \end{pmatrix}}_{\mathbf{3}} \qquad \Bigg| \qquad \underbrace{e^c}_{\mathbf{1}} \quad \underbrace{\mu^c}_{\mathbf{1}''} \quad \underbrace{\tau^c}_{\mathbf{1}'}$$

Here we consider a **see-saw realization**, so we also introduce **conjugate neutrino fields** ν^C transforming as a triplet of A_4

$$\underbrace{\begin{matrix} \begin{pmatrix} e \\ \nu_e \end{pmatrix} & \begin{pmatrix} \mu \\ \nu_\mu \end{pmatrix} & \begin{pmatrix} \tau \\ \nu_\tau \end{pmatrix} \\ \nu_{e^c} & \nu_{\mu^c} & \nu_{\tau^c} \end{matrix}}_{\mathbf{3}} \qquad \Bigg| \qquad \underbrace{e^c}_{\mathbf{1}} \quad \underbrace{\mu^c}_{\mathbf{1}''} \quad \underbrace{\tau^c}_{\mathbf{1}'}$$

We adopt a **supersymmetric (SUSY)** also to make contact with **Grand Unification**.
$\rightarrow$ Flavor symmetries are supposed to act near the GUT scale.

3. The A_4 symmetry

It is straightforward to write all of a_i, b_i and c_i elements by s and t. Then, the conjugacy classes are rewritten as

$$\begin{aligned} C_1 &: \quad e, \quad h=1, \\ C_3 &: \quad s, tst^2, t^2st, \quad h=2, \\ C_4 &: \quad t, ts, st, sts, \quad h=3, \\ C_{4'} &: \quad t^2, st^2, t^2s, tst, \quad h=3, \end{aligned}$$

Using them, we study characters. Because $s^2 = e$, the characters of C_3 have two possibilities, $\chi_\alpha(C_3) = \pm 1$.

Similarly, because of $t^3 = e$, the characters $\chi_\alpha(t)$ can correspond to three values,i.e. $\chi_\alpha(t) = \omega^n, \ n = 0, 1, 2$

Thus, all of three singlets, $\mathbf{1}, \mathbf{1}'$ and $\mathbf{1}''$ are classified by these three values of $\chi_\alpha(t) = 1, \omega$ and ω^2,respectively.

Obviously, it is found that

$$\chi_\alpha(C_{4'}) = \chi_\alpha(C_4))^2$$

Thus, the generators such as $s = a_2, t = b_1, t^2 = c_1$ are represented on the non-trivial singlets $\mathbf{1}'$ and $\mathbf{1}''$ as

$$\begin{aligned} s(\mathbf{1}') &= a_2(\mathbf{1}') = 1, \quad s(\mathbf{1}'') = a_2(\mathbf{1}'') = 1, \\ t(\mathbf{1}') &= b_1(\mathbf{1}') = \omega, \quad t(\mathbf{1}'') = b_1(\mathbf{1}'') = \omega^2, \\ t^2(\mathbf{1}') &= c_1(\mathbf{1}') = \omega^2, \quad t^2(\mathbf{1}'') = c_1(\mathbf{1}'') = \omega \end{aligned}$$

	h	χ_1	$\chi_{1'}$	$\chi_{1''}$	χ_3
C_1	1	1	1	1	3
C_3	2	1	1	1	-1
C_4	3	1	ω	ω^2	0
$C_{4'}$	3	1	ω^2	ω	0

Table: Characters of A_4

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4. The Modeling for Neutrinos and Charged-Leptons Mass

Under A_4, let the three families of leptons transform as

$$(\nu_i, l_i) \sim \underline{3}, \quad l_{iR} \sim \underline{1}, \underline{1}', \underline{1}''.$$

Add the following new particles,

$$(E^0, E^-)_{L,R} \sim \underline{1}, \; N_{L,R} \sim \underline{1}, \; s_i \sim \underline{3},$$

all assumed

- odd under Z_2 (dark) symmetry
- even under Z_2 for SM particles

where

- (E^0, E^-) : a fermion doublet
- N : a neutral fermion singlet
- s_1, s_2, s_3 : *real* neutral scalar singlets

Together with the one Higgs doublet (ϕ^+, ϕ^0) of the SM, one-loop radiative inverse seesaw neutrino masses are generated[11][12] as shown in the figure at next page.

[11] S. Fraser, E. Ma, and O. Popov, Phys. Lett. **B737**, 280 (2014)
[12] E. Ma, A. Natale, and O. Popov, Phys. Lett. **B746**, 114 (2015)

4. The Modeling for Neutrinos and Charged-Leptons Mass

Together with the one Higgs doublet (ϕ^+, ϕ^0) of the SM, one-loop radiative inverse seesaw neutrino masses are generated as shown in the below figure.

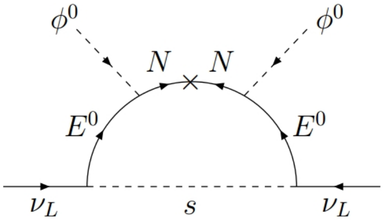

Figure: One-loop generation of inverse seesaw neutrino mass

These new terms in the Lagrangian are given by

$$\mathcal{L}' = -m_N \bar{N} N - m_E \left(\bar{E}^0 E^0 + \bar{E}^- E^- \right) - \frac{1}{2} m_L N_L N_L - \frac{1}{2} m_R N_R N_R + \frac{1}{2} (m_s^2)_{ij} s_i s_j$$
$$+ f_D \bar{N}_L \left(E_R^0 \phi^0 - E_R^- \phi^+ \right) + f_F \bar{N}_R \left(E_L^0 \phi^0 - E_L^- \phi^+ \right) + f s_i \left(\bar{E}_R^0 \nu_{iL} + \bar{E}_R^- l_{iL} \right) + H.c.$$

4. The Modeling for Neutrinos and Charged-Leptons Mass

$$\mathcal{L}' = -m_N \bar{N} N - m_E \left(\bar{E}^0 E^0 + \bar{E}^- E^-\right) - \frac{1}{2} m_L N_L N_L - \frac{1}{2} m_R N_R N_R + \frac{1}{2}(m_s^2)_{ij} s_i s_j$$
$$+ f_D \bar{N}_L \left(E_R^0 \phi^0 - E_R^- \phi^+\right) + f_F \bar{N}_R \left(E_L^0 \phi^0 - E_L^- \phi^+\right) + f s_i \left(\bar{E}_R^0 \nu_{iL} + \bar{E}_R^- l_{iL}\right) + H.c.$$

The mass matrix linking $\left(\bar{N}_L, E_L^0\right)$ to $\left(N_R, E_R^0\right)$ is then

$$\mathcal{M}_{N,E} = \begin{pmatrix} m_N & m_D \\ m_F & m_E \end{pmatrix}$$

where

- $m_D = f_D \left\langle \phi^0 \right\rangle$
- $m_F = f_F \left\langle \phi^0 \right\rangle$

As a result, N and E^0 mix to form two Dirac fermions of masses $m_{1,2}$, with mixing angles

$$m_D m_E + m_F m_N = \sin\theta_L \cos\theta_L \left(m_1^2 - m_2^2\right),$$
$$m_D m_N + m_F m_E = \sin\theta_R \cos\theta_R \left(m_1^2 - m_2^2\right).$$

and let me define field i having mass m_i for $i = 1, 2$.

4. The Modeling for Neutrinos and Charged-Leptons Mass

To connect the loop, Majorana mass terms m_L and m_R are necessary.

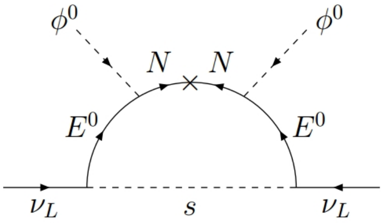

Figure: One-loop generation of inverse seesaw neutrino mass

Since both E and N may be defined to carry lepton number, these terms violate lepton number softly and may be naturally small, thus realizing the mechanism of inverse seesaw[13][14][15]

[13] D. Wyler and L. Wolfenstein, Nucl. Phys. **B218**, 205 (1983)
[14] R. N. Mohaphatra and J. W. F. Valle, Phys. Rev. **D34**, 1642 (1986)
[15] E. Ma, Phys. Lett. **B191**, 287 (1987)

4. The Modeling for Neutrinos and Charged-Leptons Mass

The one-loop Majorana neutrino mass is given by

$$
\begin{aligned}
m_\nu \;=\;& f^2 m_R \sin^2\theta_R \cos^2\theta_R (m_1^2 - m_2^2)^2 \int \frac{d^4k}{(2\pi)^4} \frac{k^2}{(k^2 - m_s^2)} \frac{1}{(k^2 - m_1^2)^2} \frac{1}{(k^2 - m_2^2)^2} \\
+\;& f^2 m_L m_1^2 \sin^2\theta_R \cos^2\theta_L \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m_s^2)} \frac{1}{(k^2 - m_1^2)^2} \\
+\;& f^2 m_L m_2^2 \sin^2\theta_L \cos^2\theta_R \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m_s^2)} \frac{1}{(k^2 - m_2^2)^2} \\
-\;& 2 f^2 m_L m_1 m_2 \sin\theta_L \sin\theta_R \cos\theta_L \cos\theta_R \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m_s^2)} \frac{1}{(k^2 - m_1^2)} \frac{1}{(k^2 - m_2^2)}
\end{aligned}
$$

This formula holds for s as a mass eigenstate.

If A_4 is unbroken, then $s_{1,2,3}$ all have the same mass and $\mathcal{M}_\nu$ is proportional to the identity matrix

If A_4 is softly broken by the necessarily real $s_i s_j$ mass terms, the ν mass matrix is given by

$$
\mathcal{M}_\nu = \mathcal{O} \begin{pmatrix} m_{\nu 1} & 0 & 0 \\ 0 & m_{\nu 2} & 0 \\ 0 & 0 & m_{\nu 3} \end{pmatrix} \mathcal{O}^T
$$

4. The Modeling for Neutrinos and Charged-Leptons Mass

If A_4 is softly broken by the necessarily real $s_i s_j$ mass terms, then the ν mass matrix is given by

$$\mathcal{M}_\nu = \mathcal{O} \begin{pmatrix} m_{\nu 1} & 0 & 0 \\ 0 & m_{\nu 2} & 0 \\ 0 & 0 & m_{\nu 3} \end{pmatrix} \mathcal{O}^T$$

where $\mathcal{O}$ is an orthogonal matrix. Now each $m_{\nu i}$ may be complex because f, m_L, m_R may be complex in ν mass equation.

Hence $\mathcal{M}_\nu$ is diagonalized by $\mathcal{O}$, which is all that is required to obtain $\theta_{23} = \pi/4$ and $\delta_{CP} = \pm\pi/2$, once U_ω is applied.

$$\begin{aligned} \mathcal{M}_\nu^{(e,\mu,\tau)} &= U^* \mathcal{M}_\nu^{(1,2,3)} U^\dagger \\ &= (U_\omega \mathcal{O})^* \mathcal{M}_\nu^{(1,2,3)} (U_\omega \mathcal{O})^\dagger \\ &= U_\omega^* \mathcal{O} \mathcal{M}_\nu^{(1,2,3)} \mathcal{O}^T U_\omega^{T*} \\ &= U_\omega^* \mathcal{M}_\nu U_\omega^{T*} \end{aligned}$$

This shows that the neutrino mass matrix does not have to be real.

It only has to be diagonalized by an orthogonal matrix.

4. The Modeling for Neutrinos and Charged-Leptons Mass

To derive U_ω, the simplest way is to copy Ref.[16] and add three Higgs doublets $\Phi_i \sim \underline{3}$. This leads to the charged-lepton mass matrix.

Before to go further, I will introduce the simplest way to derive U_ω in the mentioned Ref.

Under A_4 and L (lepton number), the color-singlet fermions and scalars of this model transform as follows.

$$\begin{aligned}
(\nu_i, l_i)_L &\sim (\underline{3}, 1), \\
l_{1R} &\sim (\underline{1}, 1), \\
l_{2R} &\sim (\underline{1}', 1), \\
l_{3R} &\sim (\underline{1}'', 1), \\
N_{iR} &\sim (\underline{3}, 0), \\
\Phi_i = \left(\phi_i^+, \phi_i^0\right) &\sim (\underline{3}, 0), \\
\eta = \left(\eta^+, \eta^0\right) &\sim (\underline{1}, -1).
\end{aligned}$$

Hence its Lagrangian has the invariant terms

$$\begin{aligned}
\mathcal{L} = \frac{1}{2} M N_{iR}^2 + f \bar{N}_{iR} \left(\nu_{iL}\eta^0 - l_{iL}\eta^+\right) \\
+ h_{ijk} \overline{(\nu_i, l_i)}_L l_{jR} \Phi_k + h.c.,
\end{aligned}$$

[16] E. Ma and G. Rajasekaran, Phys. Rev. **D64**, 113012 (2001)

$$\mathcal{L} = \frac{1}{2} M N_{iR}^2 + f \bar{N}_{iR} \left(\nu_{iL} \eta^0 - l_{iL} \eta^+ \right) + h_{ijk} \overline{(\nu_i, l_i)}_L l_{jR} \Phi_k + h.c.,$$

where

$$h_{i1k} = h_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad h_{i2k} = h_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad h_{i3k} = h_3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}$$

Thus the neutrino mass matrix (in this basis) is proportional to the unit matrix with magnitude $f^2 u^2 / M$, where $u = \langle \eta^0 \rangle$, whereas the charged-lepton mass matrix is given by

$$\mathcal{M}_l = \begin{pmatrix} h_1 v_1 & h_2 v_1 & h_3 v_1 \\ h_1 v_2 & h_2 \omega v_2 & h_3 \omega^2 v_2 \\ h_1 v_3 & h_2 \omega^2 v_3 & h_3 \omega v_3 \end{pmatrix}$$

If $v_1 = v_2 = v_3 = v$, then $\mathcal{M}_l$ is easily diagonalized:

$$U_L^\dagger \mathcal{M}_L U_R = \begin{pmatrix} \sqrt{3}h_1 v & 0 & 0 \\ 0 & \sqrt{3}h_2 v & 0 \\ 0 & 0 & \sqrt{3}h_3 v \end{pmatrix} = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix},$$

where

$$U_L = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad U_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The 6×6 Majorana mass matrix spanning $(\bar{\nu}_e, \bar{\nu}_\mu, \bar{\nu}_\tau, N_1, N_2, N_3)$ is then given by

$$\mathcal{M}_{\nu,N} = \begin{pmatrix} 0 & U_L^\dagger f u \\ U_L^* f u & M \end{pmatrix}$$

Hence the 3×3 seesaw mass matrix for $(\nu_e, \nu_\mu, \nu_\tau)$ becomes

$$\mathcal{M}_\nu = \frac{f^2 u^2}{M} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

4. The Modeling for Neutrinos and Charged-Leptons Mass

$$\mathcal{M}_\nu = \frac{f^2 u^2}{M} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

This shows that ν_μ mixes maximally with ν_τ, but since all physical neutrino masses are degenerate, there are no neutrino oscillations.

To break the degeneracy, arbitrary soft terms of the form $m_{ij} N_{iR} N_{jR}$ may be added to the below Lagrangian.

$$\mathcal{L} = \frac{1}{2} M N_{iR}^2 + f \bar{N}_{iR} \left(\nu_{iL} \eta^0 - l_{iL} \eta^+ \right) + h_{ijk} \overline{(\nu_i, l_i)}_L l_{jR} \Phi_k + m_{ij} N_{iR} N_{jR} + h.c.,$$

Then, let me go back to the original paper.

4. The Modeling for Neutrinos and Charged-Leptons Mass

To derive U_ω, the simplest way is to copy the Ref. and add three Higgs doublets $\Phi_i \sim \underline{3}$. This leads to the charged-lepton mass matrix.

$$\mathcal{M}_l = \begin{pmatrix} f_e v_1^* & f_\mu v_1^* & f_\tau v_1^* \\ f_e v_2^* & f_\mu \omega^2 v_2^* & f_\tau \omega v_2^* \\ f_e v_3^* & f_\mu \omega v_3^* & f_\tau \omega^2 v_3^* \end{pmatrix} = \begin{pmatrix} v_1^* & 0 & 0 \\ 0 & v_2^* & 0 \\ 0 & 0 & v_3^* \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \begin{pmatrix} f_e & 0 & 0 \\ 0 & f_\mu & 0 \\ 0 & 0 & f_\tau \end{pmatrix}$$

For $v_1 = v_2 = v_3$, a residual Z_3 symmetry exists with $m_e = \sqrt{3} f_e v$, etc. and U_ω becomes the transformation linking $\mathcal{M}_l$ to $\mathcal{M}_\nu$.

However, this scenario requires *four Higgs doublets*. It is thus somewhat problematic in the face of present data regarding SM Higgs.

4. The Modeling for Neutrinos and Charged-Leptons Mass

To obtain charged-lepton masses in the context of A_4 with just the SM Higgs doublet, the general radiative framework of Ref.[17] is adopted.

The specific scenario here requires the addition of two sets of charged scalars odd under dark Z_2:

$$x_i^- \sim \underline{3}, \quad y_i^- \sim \underline{1}, \underline{1}', \underline{1}''$$

The one-loop diagram is given in the below figure.

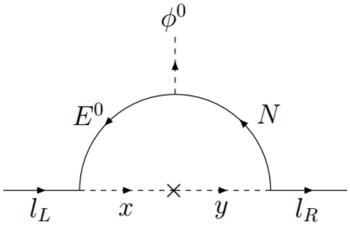

Figure: One-loop generation of charged-lepton mass

[17] E. Ma, Phys. Rev. Lett. **112**, 091801 (2014)

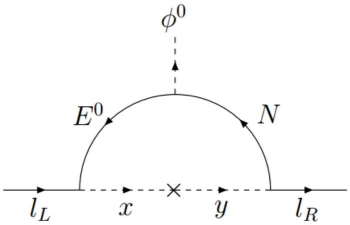

Figure: One-loop generation of charged-lepton mass

To connect x with y, A_4 must be broken, either softly

- so that the link is again U_ω to obtain the desired residual Z_3 symmetry,
- or spontaneously using three singlet scalar fields $\chi_i \sim \underline{3}$ with equal VEV.

In this way, the three Higgs doublets of the original A_4 model are replaced in a renormalizable theory for obtaining charged-lepton masses.

4. The Modeling for Neutrinos and Charged-Leptons Mass

Note that the latter may be considered as the ultraviolet completion of the common practice of using the nonrenormalizable dimension-five term $\bar{l}_L l_R \bar{\phi}^0 \chi$ for such a purpose.

Let me add detailed explanation for the above mention.

1. $\lim_{m\to 0}$ (propagator $\times$ log) $\sim \lim_{m\to 0} \frac{1}{m^2}\frac{1}{p^2-m^2}\log\left(\frac{1}{m}\right) \to \infty$
   infrared divergence

2. $\lim_{m\to \infty}$ (propagator $\times$ log) $\sim \lim_{m\to \infty} \frac{1}{m^2}\frac{1}{p^2-m^2}\log\left(\frac{1}{m}\right) \to \infty$
   ultraviolet divergence

dim-5 operator
non-renormalizable } To make the theory renormalizable

$\Uparrow$
add a term at a higher mass scale

Then, let me go back to the original paper.

4. The Modeling for Neutrinos and Charged-Leptons Mass

As a result, the charged-lepton mass matrix is given by

$$\mathcal{M}_l = U_\omega^\dagger \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix},$$

with

$$m_e = f' f_e \mu_e u \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m_{1e}^2)(k^2 - m_{2e}^2)} \left[\frac{m_1 \cos\theta_R \sin\theta_L}{k^2 - m_1^2} - \frac{m_2 \cos\theta_L \sin\theta_R}{k^2 - m_2^2} \right],$$

where

- $f' = E_L^0 l_L x^*$ Yukawa coupling
- $f_e = N_R e_R y_1^*$ Yukawa coupling
- the scalar trilinear $x y_1^* \chi$ coupling
- u is the VEV of χ
- $m_{1e,2e}$: the mass eigenvalues of the 2×2 mass-squared matrix

$$\mathcal{M}_{xy_1}^2 = \begin{pmatrix} m_x^2 & \mu_e u \\ \mu_e u & m_{y_1}^2 \end{pmatrix},$$

with $\mu_e u = \sin\theta_e \cos\theta_e \left(m_{1e}^2 - m_{2e}^2 \right)$ and similarly for m_μ and m_τ.

4. The Modeling for Neutrinos and Charged-Leptons Mass

One important consequence of a radiative charged-lepton mass is that the Higgs Yukawa coupling $h\bar{l}l$ is no longer exactly m_l/v as in the SM. Its deviation is not suppressed by the usual one-loop factor of $16\pi^2$ and may be large enough to be observable[18]

There is a one-to-one correlation of the neutrino mass eigenstates to the $s_{1,2,3}$ mass eigenstates, the lightest of which is dark matter[19][20]

It is also clear from eq. of ν mass that all three neutrino masses are expected to be of the same order of magnitude, and their mass squared differences are related to the scalar mass differences.

[18] S. Fraser and E. Ma, Europhys. Lett. **108**, 11002 (2014)
[19] V. Silveira and A. Zee, Phys. Lett. **B161**, 136 (1985)
[20] J. M. Cline, P. Scott, K. Kainulainen, and C. Weniger, Phys. Rev. **D88**, 055025 (2013)

4. The Modeling for Neutrinos and Charged-Leptons Mass

The most recent cosmological data[21] imply

$$\sum m_\nu < 0.23 \text{ eV}$$

This would mean that the effective ν mass m_{ee} in neutrinoless double decay is bounded below

- 0.07 eV for normal ordering
- 0.08 eV for inverted ordering

Due to the presence of the A_4 symmetry, the dark matter parity of this model is also derivable from lepton parity[22]. Under lepton parity, let the new particles $\left(E^0, E^-\right), N$ be even and s, x, y be odd, then the same Lagrangian is obtained.

[21] P. A. R. Ade *et al.* (PLANCK Collaboration), arXiv:1502.01589 [astro-ph.CO].
[22] E. Ma, arXiv:1502.02200 [hep-ph]

4. The Modeling for Neutrinos and Charged-Leptons Mass

As a result, dark parity is simply given by $(-1)^{L+2j}$, which is odd for all the new particles and even for all the SM particles.

Note that the tree-level Yukawa coupling $\bar{l}_L l_R \phi^0$ would be allowed by lepton parity alone, but is forbidden here because of the A_4 symmetry.

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5. Summary

In conclusion,

1. it has been pointed out that the phenomenologically successful values of $\theta_{23} = \pi/4$ and $\delta_{CP} = \pm\pi/2$ for the neutrino mixing matrix is derivable from the familiar A_4 transformation if it is multiplied by an orthogonal matrix.
2. this leads to the specific notion that a desirable neutrino mass matrix should come from three real scalars in the context of A_4.
3. to obtain the latter naturally, a specific scotogenic one-loop radiative model of neutrino and charged-lepton masses is proposed, where the particles appearing in the loop have odd dark matter parity.
4. these predicted new particles should have masses at the scale of weakly interacting dark matter, i.e. 1 TeV or less, and be potentially observable at the LHC, which has just resumed operation at CERN.