Mixing of Elementary and Composite Particles

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LECTURE – 3

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27 July 2017

Part-1

Mixing of Bosons: A General Quantum Mechanical Treatment

Setting up the problem

- **Given:** Two particles X_1 and X_2 mix with each other.
- **To find:** Phenomenological consequences of this mixing.
- **Assume:**
  - There exists some reason for mixing (e.g. decay to similar final states, similarity in compositeness).
  - Both X_1 and X_2 can arise at the place of production and get detected at the place of detection.
- **Expectations:** Effect on
  - masses,
  - total decay widths $\implies$ mean life-times,
  - individual decay rates,
  - detection probability.

Mathematical formulation of the problem

- **Assumptions:**
  - **Purity of initial state:** At production, either X_1 or X_2 is produced randomly,
  - **Common decay modes:** $X_{1,2} \to n_i$, where n_i (for $i = 1, 2, \ldots$) denotes the ith common final state,
  - **Independence of detection:** what is detected does not depend on what was produced,
  - **Sufficiently large time for propagation:** in comparison with the time scale associated with the interaction Hamiltonian for production.

- **Primary question:** We start with X_1, say, and detect effects of X_2. How is this possible? This is possible only if X_1 has oscillated to X_2.

- **QM States:** Before mixing (X_1, X_2) After mixing (X, X').
  - **Interaction via:** X_1, X_2 (no definite mass)
  - **Propagation via:** X, X' (definite mass)

Mathematical formulation of the problem

- **Orthogonal states:** $\langle X_1|X_2\rangle = 0 = \langle X|X'\rangle$
- **Normalized states:** $\langle X_1|X_1\rangle = \langle X_2|X_2\rangle = \langle X|X\rangle = \langle X'|X'\rangle = 1$
- **Mixing parametrization:** Mass eigenstates as linear combination of interaction eigenstates,

$$\left.\begin{aligned} |X\rangle &= p_1\,|X_1\rangle - q_1\,|X_2\rangle \equiv \begin{pmatrix} p_1 \\ -q_1 \end{pmatrix}, \\ |X'\rangle &= p_2\,|X_1\rangle + q_2\,|X_2\rangle \equiv \begin{pmatrix} p_2 \\ q_2 \end{pmatrix}, \end{aligned}\right\} \equiv \left\{ \begin{pmatrix} X \\ X' \end{pmatrix} = \begin{pmatrix} p_1 & -q_1 \\ p_2 & q_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \right.$$

where $p_{1,2}$ and $q_{1,2}$ are in general complex and satisfy the following,

$$\begin{aligned} &|p_1|^2 + |q_1|^2 = 1 = |p_2|^2 + |q_2|^2\,, && \text{(Normalised states)} \\ &p_1 p_2^* - q_1 q_2^* = 0, && \text{(Orthogonal states)} \\ &p_1 q_2 + p_2 q_1 \neq 0. && \text{(Invertible mixing matrix)} \end{aligned}$$

- **Decoupling (or no mixing) limit:** $X = X_1$ and $X' = X_2$ by setting $p_1 = 1 = q_2$ and $p_2 = 0 = q_1$.

Mathematical formulation of the problem

❖ **Time evolution:** At time $t = 0$ either X_1 or X_2 gets produced. After some sufficiently long time t, it is detected as either X_1 or X_2. After production the time evolution of the wavefunction must be described in terms of time evolution of the mass eigenstate, i.e.

$$|\psi(t)\rangle = \psi_1(t)\,|X\rangle + \psi_2(t)\,\big|X'\big\rangle \equiv \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},$$

and time evolution is governed by **Schrödinger-like equation**

$$i\frac{d}{dt}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = H\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},$$

where the Hamiltonian H is composed of a Hermitian part (M) and an anti-Hermitian part (Γ), responsible respectively for the mass and decay widths,

$$H \equiv M - \frac{i}{2}\Gamma = \begin{pmatrix}H_{11} & H_{12}\\ H_{21} & H_{22}\end{pmatrix} = \begin{pmatrix}M_{11} - \frac{i}{2}\Gamma_{11} & M_{12} - \frac{i}{2}\Gamma_{12}\\ M_{21} - \frac{i}{2}\Gamma_{21} & M_{22} - \frac{i}{2}\Gamma_{22}\end{pmatrix}.$$

Mathematical formulation of the problem

- Hermiticity of M and Γ matrices $\implies$ $M = M^\dagger$ (i.e. M_{11}, M_{22} are real and $M_{12} = M_{21}^*$), $\Gamma = \Gamma^\dagger$ (i.e. Γ_{11}, Γ_{22} are real and $\Gamma_{12} = \Gamma_{21}^*$).
- If we assume invariance under time reversal also, then $M_{12} = M_{21}$ and $\Gamma_{12} = \Gamma_{21}$, i.e. $H_{12} = H_{21}$.
- Thus, M_{12} and Γ_{12} (and hence M_{21} and Γ_{21}) are real. Thus, we finally have,

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} M_{11} - \frac{i}{2}\Gamma_{11} & M_{12} - \frac{i}{2}\Gamma_{12} \\ M_{12} - \frac{i}{2}\Gamma_{12} & M_{22} - \frac{i}{2}\Gamma_{22} \end{pmatrix},$$

with all the M_{ij}'s and Γ_{ij}'s real and they are defined from the self-energies for X and X'.

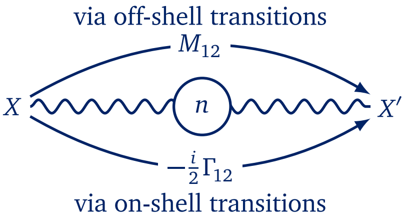
Mathematical formulation of the problem

- Considering self-energy diagrams upto 1-loop level only, in perturbation theory, we get,

$$M_{ij} = M_0\delta_{ij} + \langle i|H_I|j\rangle + \sum_n P\frac{\langle i|H_I|n\rangle\langle n|H_I|j\rangle}{M_0 - E_n},$$
$$\Gamma_{ij} = 2\pi\sum_n \delta(M_0 - E_n)\langle i|H_I|n\rangle\langle n|H_I|j\rangle,$$

where M_0 is the mass scale in which mixing is being considered, E_n is the center-of-momentum energy of the intermediate state n which connects both X and X', and the operator P projects the principal part.

Mathematical formulation of the problem



- The intermediate states that contribute to the mass are virtual (off-shell) and those contributing to the decay width are physical (on-shell).

Mathematical formulation of the problem

- The non-Hermitian nature of the Hamiltonian reflects the fact that the probability of observing either X or X' decreases with time as they are unstable and decay.
- Starting with $i\frac{d|\psi\rangle}{dt} = \left(M - \frac{i}{2}\Gamma\right)|\psi\rangle$, we have

$$\begin{aligned}\frac{d}{dt}|\psi|^2 \equiv \frac{d}{dt}\langle\psi|\psi\rangle &= \langle\dot{\psi}|\psi\rangle + \langle\psi|\dot{\psi}\rangle \\ &= -\langle\psi|\Gamma/2 - iM|\psi\rangle - \langle\psi|\Gamma/2 + iM|\psi\rangle \\ &= -\langle\psi|\Gamma|\psi\rangle.\end{aligned}$$

- Thus

$$\frac{d}{dt}\left(|\psi_1|^2 + |\psi_2|^2\right) = -\begin{pmatrix}\psi_1^* & \psi_2^*\end{pmatrix}\begin{pmatrix}\Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22}\end{pmatrix}\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}.$$

- This implies that for any ψ_1 and ψ_2, the LHS above must always be negative. In other words, the decay width matrix Γ must be positive definite, i.e. $\Gamma_{11} \geqslant 0$, $\Gamma_{22} \geqslant 0$ and $\det\Gamma \geqslant 0$.

Mathematical formulation of the problem

With these mathematical details, our problem is to find out phenomenologically significant effects due to the mixing of X and X'.

Finding phenomenological consequences of mixing

- The eigenvectors of H give the two mass eigenstates X and X'.
- Let M_X, Γ_X respectively denote the experimentally observed mass and decay width of X. Similarly, $M_{X'}$ and $\Gamma_{X'}$ respectively denote the mass and decay width of X'. The masses M_X, $M_{X'}$ and the decay widths Γ_X, $\Gamma_{X'}$ are positive real quantities.
- Assume that $M_{X'} > M_X$. The eigenvalues of the Hamiltonian can be obtained as follows:

$$\begin{vmatrix} H_{11} - \mu & H_{12} \\ H_{12} & H_{22} - \mu \end{vmatrix} = 0 \quad (\because H_{21} = H_{12})$$

$$\implies \mu = \frac{H_{11} + H_{22} \pm \sqrt{(H_{22} - H_{11})^2 + 4H_{12}^2}}{2}.$$

Finding phenomenological consequences of mixing

- Let us denote the two eigenvalues by μ_X and $\mu_{X'}$ as follows:

$$\mu_X \equiv M_X - \frac{i}{2}\Gamma_X = \frac{H_{11}+H_{22}}{2} - \sqrt{\left(\frac{H_{22}-H_{11}}{2}\right)^2 + H_{12}^2},$$

$$\mu_{X'} \equiv M_{X'} - \frac{i}{2}\Gamma_{X'} = \frac{H_{11}+H_{22}}{2} + \sqrt{\left(\frac{H_{22}-H_{11}}{2}\right)^2 + H_{12}^2},$$

such that

$$\Delta\mu = \mu_{X'} - \mu_X = \sqrt{(H_{22}-H_{11})^2 + 4H_{12}^2}.$$

- From these we get

$$M_X + M_{X'} = M_{11} + M_{22} \ ,$$

$$\Gamma_X + \Gamma_{X'} = \Gamma_{11} + \Gamma_{22} \ .$$

Finding phenomenological consequences of mixing

- Define a quantity ϑ as follows:

$$\vartheta \equiv \frac{H_{22} - H_{11}}{\Delta\mu},$$

such that

$$\Delta\mu^2(1-\vartheta^2) = 4H_{12}^2.$$

- The two eigenvectors corresponding to the eigenvalues μ_X and $\mu_{X'}$ satisfy the following equations:

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix} \begin{pmatrix} p_1 \\ -q_1 \end{pmatrix} = \mu_X \begin{pmatrix} p_1 \\ -q_1 \end{pmatrix},$$

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix} \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = \mu_{X'} \begin{pmatrix} p_2 \\ q_2 \end{pmatrix}.$$

- From above

$$\frac{q_1}{p_1} = \frac{H_{11} - \mu_X}{H_{12}} = \frac{H_{12}}{H_{22} - \mu_X} \implies \frac{q_1}{p_1} = \frac{\Delta\mu(1-\vartheta)}{2H_{12}} = \frac{2H_{12}}{\Delta\mu(1+\vartheta)},$$

$$\frac{q_2}{p_2} = \frac{\mu_{X'} - H_{11}}{H_{12}} = \frac{H_{12}}{\mu_{X'} - H_{22}} \implies \frac{q_2}{p_2} = \frac{\Delta\mu(1+\vartheta)}{2H_{12}} = \frac{2H_{12}}{\Delta\mu(1-\vartheta)}.$$

Finding phenomenological consequences of mixing

- It is now easy to show that

$$\frac{q_2}{p_2}\frac{q_1}{p_1} = 1, \quad \text{and} \quad \vartheta = \frac{q_2/p_2 - q_1/p_1}{q_2/p_2 + q_1/p_1}.$$

- In general, $H_{22} \neq H_{11}$, which implies that $\vartheta \neq 0 \implies \frac{q_2}{p_2} \neq \frac{q_1}{p_1}$.
- Let us define another quantity

$$\zeta = \frac{q_2/p_2}{q_1/p_1} = \frac{q_2}{q_1}\frac{p_1}{p_2} = \frac{1-\vartheta}{1+\vartheta}.$$

Since, in our case $\vartheta \neq 0$, we have $\zeta \neq 1$.

- If we take $z \equiv (\Delta\mu)^2$, then it can be easily shown that

$$M_X = \frac{M_{11}+M_{22}}{2} - \frac{1}{2}\sqrt{\frac{|z|+\Re(z)}{2}},$$

$$\Gamma_X = \frac{\Gamma_{11}+\Gamma_{22}}{2} + \sqrt{\frac{|z|-\Re(z)}{2}}.$$

Finding phenomenological consequences of mixing

- Doing some more algebra we get the following,

$$\Gamma_{12}^2 = \frac{1}{2}\left(\mathsf{A} \pm \sqrt{\mathsf{A}^2 + 4\mathsf{B}}\right),$$
$$M_{12} = -\frac{\sqrt{\mathsf{B}}}{2\Gamma_{12}},$$

  where

$$\mathsf{A} = \left(\Gamma_{11}\Gamma_{22} + \Gamma_X\Gamma_{X'}\right) - 4\left(M_{11}M_{22} + M_X M_{X'}\right),$$
$$\mathsf{B} = \frac{1}{4}\left[\pm\left(M_{X'} - M_X\right)\left(\Gamma_{X'} - \Gamma_X\right) + \left(M_{22} - M_{11}\right)\left(\Gamma_{22} - \Gamma_{11}\right)\right]^2.$$

- Provided we know the full theory, $M_{11}, M_{22}, \Gamma_{11}$ and Γ_{22} can be computed from self-energy diagrams. Then we can predict for Γ_{12} which can be verified experimentally.

Finding phenomenological consequences of mixing

- There exists another way of parametrizing the Hamiltonian using the Pauli σ matrices:

$$\mathbf{H} \equiv \mathbf{M} - \frac{i}{2}\Gamma = E_1\sigma_1 + E_2\sigma_2 + E_3\sigma_3 - iD\mathbf{1},$$

where E_1, E_2, E_3 and D are complex quantities, and σ_1, σ_2, σ_3 are the Pauli σ matrices and $\mathbf{1}$ is the 2×2 identity matrix. Therefore

$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} E_3 - iD & E_1 - iE_2 \\ E_1 + iE_2 & -E_3 - iD \end{pmatrix},$$

which implies that

$$E_1 = \frac{H_{12}+H_{21}}{2} = \frac{M_{12}+M_{21}}{2} - \frac{i(\Gamma_{12}+\Gamma_{21})}{4},$$

$$E_2 = \frac{i(H_{12}-H_{21})}{2} = \frac{i(M_{12}-M_{21})}{2} + \frac{\Gamma_{12}-\Gamma_{21}}{4},$$

$$E_3 = \frac{H_{11}-H_{22}}{2} = \frac{M_{11}-M_{22}}{2} - \frac{i(\Gamma_{11}-\Gamma_{22})}{4},$$

$$D = \frac{i(H_{11}+H_{22})}{2} = \frac{i(M_{11}+M_{22})}{2} + \frac{\Gamma_{11}+\Gamma_{22}}{4}.$$

Finding phenomenological consequences of mixing

- It is convenient to write

$$E_1 = E \sin\theta \cos\phi,$$
$$E_2 = E \sin\theta \sin\phi,$$
$$E_3 = E \cos\theta,$$

where $E = \sqrt{E_1^2 + E_2^2 + E_3^2}$, θ and ϕ are, in general, complex. It is easy to see that

$$E = \frac{1}{2}\sqrt{(H_{22} - H_{11})^2 + 4H_{12}H_{21}} = \frac{\Delta\mu}{2}.$$

- The Hamiltonian now takes the following form

$$\mathbf{H} = \begin{pmatrix} E\cos\theta - iD & e^{-i\phi}E\sin\theta \\ e^{i\phi}E\sin\theta & -E\cos\theta - iD \end{pmatrix}.$$

Finding phenomenological consequences of mixing

- The eigenvalues of the Hamiltonian are now obtained as follows

$$\begin{vmatrix} E\cos\theta - iD - \mu & e^{-i\phi}E\sin\theta \\ e^{i\phi}E\sin\theta & -E\cos\theta - iD - \mu \end{vmatrix} = 0$$

$$\implies \mu = -iD \pm E.$$

Therefore, by comparison with the eigenvalues, we get

$$\mu_{X'} = -iD + E,$$
$$\mu_X = -iD - E.$$

- It is now easy to notice that

$$\vartheta = \frac{-2E_3}{\Delta\mu} = \frac{-2E\cos\theta}{2E} = -\cos\theta.$$

Finding phenomenological consequences of mixing

- Recollect that in general $\vartheta \neq 0$, which implies that $\theta \neq \pi/2$.
- It is easy to show the following,

$$\frac{q_2}{p_2} = e^{i\phi} \tan\frac{\theta}{2},$$
$$\frac{q_1}{p_1} = e^{i\phi} \cot\frac{\theta}{2}.$$

Thus

$$\frac{q_2}{p_2}\frac{q_1}{p_1} = \frac{H_{21}}{H_{12}} = \frac{M_{12}^* - \frac{i}{2}\Gamma_{12}^*}{M_{12} - \frac{i}{2}\Gamma_{12}} = e^{2i\phi}.$$

- If we consider q_1/p_1 and q_2/p_2 to be real, then

$$\Im(\theta) = 0 \quad \text{and} \quad \Re(\phi) = 0,$$
$$\Re(\theta) \equiv \theta_M \quad \text{and} \quad \Im(\phi) \equiv \phi_M,$$
$$\implies \frac{q_2}{p_2} = e^{-\phi_M} \tan\frac{\theta_M}{2} \quad \text{and} \quad \frac{q_1}{p_1} = e^{-\phi_M} \cot\frac{\theta_M}{2}.$$

Thus

$$\frac{q_2}{p_2}\frac{q_1}{p_1} = \frac{H_{21}}{H_{12}} = e^{-2\phi_M}.$$

Finding phenomenological consequences of mixing

- The X-X' mixing matrix can thus be described by two real parameters θ_M and ϕ_M:

$$\begin{pmatrix} X \\ X' \end{pmatrix} = \begin{pmatrix} N_2 \sin(\theta_M/2) & -N_2 e^{-\phi_M} \cos(\theta_M/2) \\ N_1 \cos(\theta_M/2) & N_1 e^{-\phi_M} \sin(\theta_M/2) \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix},$$

where

$$N_1 = \frac{1}{\sqrt{\cos^2(\theta_M/2) + e^{-2\phi_M} \sin^2(\theta_M/2)}},$$

$$N_2 = \frac{1}{\sqrt{\sin^2(\theta_M/2) + e^{-2\phi_M} \cos^2(\theta_M/2)}}.$$

Finding phenomenological consequences of mixing

If we put $\phi_M = 0$, then

$$\begin{pmatrix} X \\ X' \end{pmatrix} = \begin{pmatrix} \sin(\theta_M/2) & -\cos(\theta_M/2) \\ \cos(\theta_M/2) & \sin(\theta_M/2) \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix},$$

which is the most usual mixing marix we encounter.

In the next lectue we shall study some more consequences, notably CP violation.