Mixing of Elementary and Composite Particles

Dibyakrupa Sahoo

LECTURE – 4

Department of Physics, and
Institute of Physics and Applied Physics,
Yonsei University, Seoul, South Korea

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Part-2

Mixing of Bosons: Some Consequences and Examples

Recap of Part-1

Mixing of Bosons: A General Quantum Mechanical Treatment

Propagation X, X'

Interaction X_1, X_2

$$H \equiv M - \frac{i}{2}\Gamma = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} M_{11} - \frac{i}{2}\Gamma_{11} & M_{12} - \frac{i}{2}\Gamma_{12} \\ M_{21} - \frac{i}{2}\Gamma_{21} & M_{22} - \frac{i}{2}\Gamma_{22} \end{pmatrix}$$

$$\begin{pmatrix} X \\ X' \end{pmatrix} = \begin{pmatrix} p_1 & -q_1 \\ p_2 & q_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

$M = M^\dagger$
$\Gamma = \Gamma^\dagger$
T invariance

Eigenvalues

$$M_{ij} = M_0 \delta_{ij} + \langle i | H_I | j \rangle + \sum_n P \frac{\langle i | H_I | n \rangle \langle n | H_I | j \rangle}{M_0 - E_n}$$

$$\Gamma_{ij} = 2\pi \sum_n \delta\left(M_0 - E_n\right) \langle i | H_I | n \rangle \langle n | H_I | j \rangle$$

where $i, j \in \{X, X'\}$

$$\mu_{X,X'} = \frac{H_{11} + H_{22} \mp \Delta\mu}{2}$$

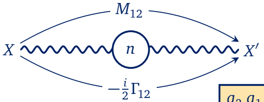
$$\Delta\mu = \sqrt{\left(H_{22} - H_{11}\right)^2 + 4H_{12}^2}$$

$$\vartheta = \left(H_{22} - H_{11}\right) / \Delta\mu$$

$$\frac{q_1}{p_1} = \frac{\Delta\mu(1-\vartheta)}{2H_{12}} = \frac{2H_{12}}{\Delta\mu(1+\vartheta)}$$

$$\frac{q_2}{p_2} = \frac{\Delta\mu(1+\vartheta)}{2H_{12}} = \frac{2H_{12}}{\Delta\mu(1-\vartheta)}$$

via off-shell transitions



via on-shell transitions

$$\frac{q_2}{p_2}\frac{q_1}{p_1} = 1, \quad \vartheta = \frac{q_2/p_2 - q_1/p_1}{q_2/p_2 + q_1/p_1}$$

$$\begin{pmatrix} X \\ X' \end{pmatrix} = \begin{pmatrix} \sin\left(\theta_M/2\right) & -\cos\left(\theta_M/2\right) \\ \cos\left(\theta_M/2\right) & \sin\left(\theta_M/2\right) \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

Using spherical coordinates

Continuing from Part-1

Mixing angle in terms of masses and decay widths

- The mixing angle θ_M is real, and

$$\tan^2(\theta_M/2) = \frac{q_2^2}{p_2^2} = \frac{1+\vartheta}{1-\vartheta},$$

where

$$\vartheta = \frac{H_{22}-H_{11}}{\Delta\mu} = \frac{M_{22}-M_{11}}{M_{X'}-M_X} = \frac{\Gamma_{22}-\Gamma_{11}}{\Gamma_{X'}-\Gamma_X}.$$

- It is easy to show that

$$\boxed{\tan^2(\theta_M/2) = \frac{M_{X'}-M_{11}}{M_{11}-M_X} = \frac{\Gamma_{X'}-\Gamma_{11}}{\Gamma_{11}-\Gamma_X}.}$$

- In the **decoupling (or no mixing) limit** we have $\theta_M = \pi$, which implies that $M_X = M_{11}$, and $\Gamma_X = \Gamma_{11}$, as they should be.

Prediction for mass of X'

- From the expression for the mixing angle, we have

$$M_{X'} = \left(\frac{M_X \Gamma_{11} - M_{11} \Gamma_X}{\Gamma_{11} - \Gamma_X} \right) + \Gamma_{X'} \left(\frac{M_{11} - M_X}{\Gamma_{11} - \Gamma_X} \right).$$

- If X' were a narrow resonance, then $\frac{\Gamma_{X'}}{M_{X'}} \to 0$.
- Thus, for a narrow resonance X' which mixes with another resonance X of mass M_X and decay width Γ_X, we have

$$M_{X'} \to \frac{M_X \Gamma_{11} - M_{11} \Gamma_X}{\Gamma_{11} - \Gamma_X}.$$

- **Note:** When $\Gamma_{11} = \Gamma_X$, then $M_{X'} \to \infty$, i.e. the X' becomes so heavy that it is impossible for X and X' to mix, and hence $\theta_M \to \pi$. *Looks odd to have such large mixing angle. But only looking at angle is not correct.*

Connecting θ_M with the usual mixing angle in literature

- The usual X-X' mixing is described as follows in the literature,

$$\begin{pmatrix} X \\ X' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}.$$

- Angles θ and $\theta_M/2$ differ by a constant phase only,

$$\theta = (\pi - \theta_M)/2.$$

- So the decoupling (or no mixing) limit now corresponds to $\theta = 0$.
- And the maximal mixing corresponds to $\theta = \theta_M/2 = \pi/4$.

Usual result for mixing in the literature

- **Starting point:**
  - Mixing matrix ($\equiv$ rotation matrix joining the two bases):

$$\begin{pmatrix} X \\ X' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}.$$

  - Effective Lagrangian in interaction basis:

$$\mathscr{L}_{\text{eff}} \supset m_{11}^2 (X_1 \cdot X_1) + m_{12}^2 (X_1 \cdot X_2) + m_{21}^2 (X_2 \cdot X_1) + m_{22}^2 (X_2 \cdot X_2),$$

  where

$$X_i \cdot X_j = \begin{cases} X_i^* X_j & \big(\text{when } X_i, X_j \text{ are spin-0 bosons}\big) \\ X_i^{*\mu} X_{j,\mu} & \big(\text{when } X_i, X_j \text{ are spin-1 bosons}\big) \end{cases}.$$

  - Effective Lagrangian in mass (propagation) basis:

$$\mathscr{L}_{\text{eff}} \supset M_X^2 (X \cdot X) + M_{X'}^2 \left(X' \cdot X'\right),$$

  where the dot product has the same meaning as above.

- **Note:** $M_{ij} \neq m_{ij}$.

Usual result for mixing in the literature

- **Important steps:**
  - Mass-square matrices, $m = \begin{pmatrix} m_{11}^2 & m_{12}^2 \\ m_{21}^2 & m_{22}^2 \end{pmatrix}$ and $m_D = \begin{pmatrix} M_X^2 & 0 \\ 0 & M_{X'}^2 \end{pmatrix}$.
  - The mass-square matrices m and m_D are related by a *general similarity transformation:*

$$m_D = R^T \, m \, R \Longleftrightarrow m = R \, m_D \, R^T,$$

where $R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

Usual result for mixing in the literature

❖ **Consequence:**

$$\underbrace{\begin{pmatrix} m_{11}^2 & m_{12}^2 \\ m_{21}^2 & m_{22}^2 \end{pmatrix}}_{m} = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{R} \underbrace{\begin{pmatrix} M_X^2 & 0 \\ 0 & M_{X'}^2 \end{pmatrix}}_{m_D} \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{R^T}$$

$$= \begin{pmatrix} M_X^2 \cos^2\theta + M_{X'}^2 \sin^2\theta & \left(M_{X'}^2 - M_X^2\right)\cos\theta\sin\theta \\ \left(M_{X'}^2 - M_X^2\right)\cos\theta\sin\theta & M_X^2 \sin^2\theta + M_{X'}^2 \cos^2\theta \end{pmatrix}$$

Therefore

$$m_{11}^2 + m_{22}^2 = M_X^2 + M_{X'}^2, \quad m_{22}^2 - m_{11}^2 = \left(m_{X'}^2 - M_X^2\right)\cos 2\theta,$$

and

$$\boxed{\sin^2\theta = \frac{m_{11}^2 - M_X^2}{M_{X'}^2 - M_X^2} = \frac{M_{X'}^2 - m_{22}^2}{M_{X'}^2 - M_X^2}.}$$

A classic example of meson mixing: ω-ϕ mixing

The $SU(3)$ flavor symmetry and Gell-Mann–Okubo mass formula

- The quarks u, d and s form the fundamental representation of $SU(3)$ flavor symmetry, using which all light hadrons (mesons and baryons) can be constructed, and their properties also can be well explained.
- $SU(3)$ flavor symmetry is not exact. It is broken because of the non-degenerate masses of u, d and s.
- The Gell-Mann–Okubo mass formula are set of *phenomenological* formula relating masses of light mesons and baryons in the same $SU(3)$ flavor multiplet. These formula are valid within percent level.
  - 0^- meson octet: $\underbrace{4m_K^2}_{\approx 0.98\ \text{GeV}^2} = \underbrace{m_\pi^2 + 3m_\eta^2}_{\approx 0.92\ \text{GeV}^2}$
  - $1/2^+$ baryon octet: $\underbrace{\frac{m_\Sigma + 3m_\Lambda}{2}}_{\approx 2.23\ \text{GeV}} = \underbrace{m_N + m_\Xi}_{\approx 2.25\ \text{GeV}}$
  - $3/2^+$ baryon decouplet: $\underbrace{m_\Omega - m_{\Xi^*}}_{137\ \text{MeV}} = \underbrace{m_{\Xi^*} - m_{\Sigma^*}}_{148\ \text{MeV}} = \underbrace{m_{\Sigma^*} - m_\Delta}_{155\ \text{MeV}}$

A classic example of meson mixing: ω-ϕ mixing

- Consider the two isospin $I = 0$ and strangness $S = 0$ states belonging to the 1^- vector meson octet and singlet, which are denoted as follows;

$$\omega = \frac{1}{\sqrt{6}}\left(u\bar{u} + d\bar{d} - 2s\bar{s}\right), \qquad \text{(octet)}$$
$$\phi = \frac{1}{\sqrt{3}}\left(u\bar{u} + d\bar{d} + s\bar{s}\right). \qquad \text{(singlet)}$$

- Just like 0^- meson octet case we have Gell-Mann–Okubo mass formula for this also: $3m_\omega^2 = 4m_{K^*}^2 - m_\rho^2$. From this the predicted mass for ω is $m_\omega^{\text{GMO}} = 931$ MeV, but experimentally $m_\omega^{\text{exp}} = 783$ MeV. The mass of ϕ which has the same quantum numbers $(I = 0,\ S = 0)$ as ω is experimentally found to be $m_\phi^{\text{exp}} = 1020$ MeV.
- So the idea of octet-singlet mixing (or ω-ϕ) mixing is introduced to explain the mass difference between prediction and experimental data.

A classic example of meson mixing: ω-ϕ mixing

- Step 1: Change notation to octet and singlet vector mesons.

$$V_8 = \frac{1}{\sqrt{6}}\left(u\bar{u} + d\bar{d} - 2s\bar{s}\right), \qquad \text{(octet)}$$
$$V_1 = \frac{1}{\sqrt{3}}\left(u\bar{u} + d\bar{d} + s\bar{s}\right). \qquad \text{(singlet)}$$

- Step 2: Write ω and ϕ as a linear combination of V_8 and V_1.

$$\begin{pmatrix} \omega \\ \phi \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} V_8 \\ V_1 \end{pmatrix},$$

where θ is the mixing angle.

- Step 3: Write mass-square matrices.

$$m = \begin{pmatrix} m_{88}^2 & m_{81}^2 \\ m_{18}^2 & m_{11}^2 \end{pmatrix}, \quad m_D = \begin{pmatrix} m_\omega^2 & 0 \\ 0 & m_\phi^2 \end{pmatrix},$$

where m_{88}^2 is already predicted from the Gell-Mann–Okubo mass formula, while m_ω and m_ϕ are measured experimentally.

A classic example of meson mixing: ω-ϕ mixing

❖ Step 4: Connect m_D and m by a similarity transformation.

$$m = R\, m_D\, R^T, \text{ where } R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

❖ Step 5: Get a relationship between m_{88}, m_ω, m_ϕ and θ.

$$\sin^2\theta = \frac{m_{88}^2 - m_\omega^2}{m_\phi^2 - m_\omega^2}.$$

❖ **Consequences:**

- ✡ $\sin\theta = 0.79$ which is very close to $\sqrt{(2/3)} = 0.81$. If $\sin\theta = \sqrt{(2/3)}$, then $\omega = \left(u\bar{u} + d\bar{d}\right)/\sqrt{2}$ and $\phi = s\bar{s}$. This is called *ideal mixing*.
- ✡ Since the mixing is close to the ideal mixing, ω has predominantly non-strange quarks and ϕ has mostly strange quark composition.

A comparison of results

Considering both mass and decay width

$$\begin{pmatrix} X \\ X' \end{pmatrix} = \begin{pmatrix} \sin\left(\frac{\theta_M}{2}\right) & -\cos\left(\frac{\theta_M}{2}\right) \\ \cos\left(\frac{\theta_M}{2}\right) & \sin\left(\frac{\theta_M}{2}\right) \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

$$\tan^2(\theta_M/2) = \frac{M_{X'} - M_{11}}{M_{11} - M_X} = \frac{\Gamma_{X'} - \Gamma_{11}}{\Gamma_{11} - \Gamma_X}$$

Note: Relates both the masses and decay widths of X and X' to the mixing angle.

Considering mass-square matrix alone

$$\begin{pmatrix} X \\ X' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

$$\sin^2\theta = \frac{m_{11}^2 - M_X^2}{M_{X'}^2 - M_X^2} = \frac{M_{X'}^2 - m_{22}^2}{M_{X'}^2 - M_X^2}$$

Note: Does not say anything about how the decay widths of X and X' are related.

$$\theta_M = \pi - 2\theta.$$

$$M_{ij} \neq m_{ij}.$$

How to find the dependence of decay width on mixing angle, in the second scenario?

- The decay width can be found, in the second scenario, only if we know the interaction Lagrangian. So it is a model dependent calculation.
- The decay width after mixing must give us back the decay width before mixing in the decoupling limit.
- To see how we can find out decay width in this case let us consider one example, the Z-Z' mixing.
- When we consider the mixing of Z and Z', the mass eigenstates Z and Z' which are experimentally observed are different from the interaction states Z_1 and Z_2. *Only in the decoupling (or no mixing) limit*, the interaction states are the same as the mass eigenstates $Z \equiv Z_1$ and $Z' \equiv Z_2$.

Example: Effect of Z-Z' mixing on Z decay width

- Mixing matrix:

$$\begin{pmatrix} Z \\ Z' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$$

- In the SM we know how Z_1 couples to the fermions. Now we assume that Z_2 couples to all the SM fermions in a similar way as Z_1 does albeit with a different strength:

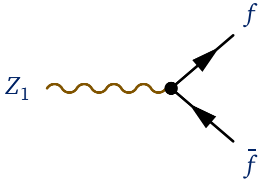
$$\begin{aligned} \mathscr{L}_{NC} = & -i\frac{g_{Z_1}}{2}\sum_f \bar{f}\gamma^\mu\left(v_1^f - a_1^f\gamma^5\right) f Z_{1\mu} \\ & -i\frac{g_{Z_2}}{2}\sum_f \bar{f}\gamma^\mu\left(v_2^f - a_2^f\gamma^5\right) f Z_{2\mu}, \end{aligned}$$

where g_{Z_1} and g_{Z_2} are the weak coupling constants, the coefficients v_1^f, a_1^f, v_2^f and a_2^f are the vector and axial vector couplings. Only those terms which are related to Z_1 are known from SM.

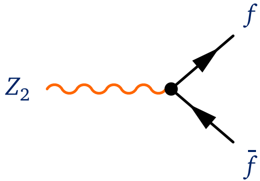
Example: Effect of Z-Z' mixing on Z decay width

Vertex factors

- Vertex factors



$$= -i\frac{g_{Z_1}}{2}\gamma^\mu\left(v_1^f - a_1^f\,\gamma^5\right),$$



$$= -i\frac{g_{Z_2}}{2}\gamma^\mu\left(v_2^f - a_2^f\,\gamma^5\right).$$

- Only those terms which are related to Z_1 are known,

$$g_{Z_1} = \frac{e}{\sin\theta_W\,\cos\theta_W}, \qquad \left(e = \sqrt{4\pi\alpha}\right)$$

where θ_W is the weak mixing angle.

Example: Effect of Z-Z' mixing on Z decay width

Vertex factors

- The vector and axial vector couplings for different fermions f coming from decay of Z_1 are given below.

fermion f	v_1^f	a_1^f
ν_e, ν_μ, ν_τ	$\frac{1}{2}$	$\frac{1}{2}$
e^-, μ^-, τ^-	$-\frac{1}{2} + 2\sin^2\theta_W$	$-\frac{1}{2}$
u, c, t	$\frac{1}{2} - \frac{4}{3}\sin^2\theta_W$	$\frac{1}{2}$
d, s, b	$-\frac{1}{2} + \frac{2}{3}\sin^2\theta_W$	$-\frac{1}{2}$

- We do not know values for any of the terms related to Z_2.

Example: Effect of Z-Z' mixing on Z decay width

Rewriting the Lagrangian

❖ We can write the Lagrangian $\mathscr{L}_{NC}$ in terms of the massive fields Z and Z' as well,

$$\mathscr{L}_{NC} = -i\,\frac{g_Z}{2}\sum_f \bar{f}\,\gamma^\mu\left(v^f - a^f\,\gamma^5\right) f\,Z_\mu - i\,\frac{g_{Z'}}{2}\sum_f \bar{f}\,\gamma^\mu\left(v^{f\prime} - a^{f\prime}\,\gamma^5\right) f\,Z'_\mu,$$

where $g_Z = g_{Z_1}$, $g_{Z'} = g_{Z_2}$,

$$v^f = v_1^f \cos\theta + \left(\frac{g_{Z'}}{g_Z}\right) v_2^f \sin\theta,$$

$$a^f = a_1^f \cos\theta + \left(\frac{g_{Z'}}{g_Z}\right) a_2^f \sin\theta,$$

$$v^{f\prime} = v_2^f \cos\theta - \left(\frac{g_Z}{g_{Z'}}\right) v_1^f \sin\theta,$$

$$a^{f\prime} = a_2^f \cos\theta - \left(\frac{g_Z}{g_{Z'}}\right) a_1^f \sin\theta.$$

$$\begin{pmatrix} g_Z v^f \\ g_{Z'} v^{f\prime} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} g_Z v_1^f \\ g_{Z'} v_2^{f\prime} \end{pmatrix},$$

$$\begin{pmatrix} g_Z a^f \\ g_{Z'} a^{f\prime} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} g_Z a_1^f \\ g_{Z'} a_2^{f\prime} \end{pmatrix}.$$

Example: Effect of Z-Z' mixing on Z decay width

Decay width of Z

- The decay rate for $Z \to f\bar{f}$, where f is any quark or lepton, is given by

$$\Gamma\left(Z \to f\bar{f}\right) = \frac{g_Z^2 M_Z}{48\pi}\left(\left(v^f\right)^2 + \left(a^f\right)^2\right).$$

- Due to Z-Z' mixing this gets modified to the following,

$$\begin{aligned}\Gamma\left(Z \to f\bar{f}\right) = \frac{g_Z^2 M_Z}{48\pi}\Bigg[&\left(\left(v_1^f\right)^2 + \left(a_1^f\right)^2\right)\cos^2\theta \\ &+ \left(\frac{g_{Z'}}{g_Z}\right)^2\left(\left(v_2^f\right)^2 + \left(a_2^f\right)^2\right)\sin^2\theta \\ &+ \left(\frac{g_{Z'}}{g_Z}\right)\left(v_1^f v_2^f + a_1^f a_2^f\right)\sin 2\theta\Bigg].\end{aligned}$$

- In the **decoupling (no mixing) limit**, $\theta = 0$ and the $\Gamma\left(Z \to f\bar{f}\right)$ remains unchanged from its SM predicted value.

Next Lecture: CP violation and mixing

- The concept of CP violation in neutral meson decays is intimately tied up with the idea of neutral meson mixings.
- Since this is a beautiful and important concept, I will go through it in detail in the next lecture.
- In the next lecture I shall also revise some basic concepts so that the lecture is self-sufficient.