EXTRA SPACETIME DIMENSIONS:
General Properties
Original Kaluza-Klein Idea

• Why consider physics in more than four dimension?
  → Kaluza-Klein: If extra dimensions exist, then it is possible to unify Gravity and EM Forces.

• Extra dimensions provide a way of unifying particles with different spins.

• Since the D-dimensional Lorentz group SO (D-1, 1) is larger than the four-dimensional Lorentz group SO(3, 1) for all D > 4, each representation of the D-dimensional Lorentz group is larger than the corresponding representation of the four-dimensional Lorentz group.
A single representation of the D-dimensional Lorentz group decomposes into several different representations of the four-dimensional Lorentz group all at once. However, different representation correspond to different physical particles of different spins.
• The case originally considered by Kaluza and Klein

\[ g^{(5)}_{\mu \nu} = \begin{pmatrix}
  g_{00} & g_{01} & g_{02} & g_{03} & g_{04} \\
  g_{11} & g_{12} & g_{13} \\
  g_{22} & g_{23} \\
  g_{33} & g_{34} \\
  g_{44}
\end{pmatrix} \]

\[ g^{(4)}_{\mu \nu} \Phi \]

\[ g_{\mu 4} = A_\mu \]

4D Brans–Dicke gravity

4D photon (E&M)
In such a theory, the fundamental field is the five-dimensional metric $g^{(5)}_{MN}$, with $(M, N) = 0, 1, 2, 3, 4$. As a symmetric tensor field, $g^{(5)}_{MN}$ has 15 independent components. Decomposed into representations of the four-dimensional Lorentz group, however, we find that $g^{(5)}_{MN}$ yields three different representations:

1. A spin-two symmetric tensor field $g^{(4)}_{\mu\nu} = g^{(5)}_{\mu\nu}$ with $0 \leq \mu, \nu \leq 3$;

2. A spin-one massless vector field $A^{(4)}_{\mu} = g^{(5)}_{4\mu}$ with $0 \leq \mu \leq 3$; and

3. A spin-zero massless scalar field $\Phi = g^{(5)}_{44}$. 

• Now we know that there are four fundamental forces: Gravity, EM, The Weak Nuclear Forces, and The Strong Nuclear Forces. Thus, unification a’la Kaluza-Klein would require even more extra dimensions.
The Compactification Procedure

• In our everyday world we are not familiar with any additional space dimensions.
• If any extra space dimensions exist, they must be sufficiently small size that they are not observable with the naked eye. In other words, they must be compactified.
• What does it mean to compactify an extra dimension?
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→ Imagine that our extra dimension takes some compact shape, such as circle, such that by travelling along that extra dimension we quickly return to our original location.
Thus, taking a circle compactification as our example, we imagine that at every point in spacetime there exists an additional circle of radius $R$ which is orthogonal to all of the known dimensions.
• In this example, spacetime has the topology \( \mathcal{M}_4 \times S_1 \).

• More generally, we can compactify \( \delta \) extra spacetime dimensions by formulating a theory on \( \mathcal{M}_4 \times K \).
• The fact that $K$ is compact means that we must impose Boundary Conditions appropriate for the specific choice of $K$.

• Let $x^\mu$ indicate the coordinates on $\mathcal{M}_4$ along the infinite real line, and let $y^i$ indicate the coordinates of $K$.

• For our example, compactification on a circle then requires that we impose periodic boundary conditions under $y \rightarrow y + 2\pi R$. 
• Construct suitable wavefunction $\Phi(x^\mu, y)$ on this extra space dimension, for our case

$$\Phi(x^\mu, y) = \frac{1}{\sqrt{2\pi R}} \sum_{n=-\infty}^{\infty} \phi_n(x^\mu) \exp(iny / R).$$

• Let us imagine that the five-dimensional field $\Phi(x^\mu, y)$ is a complex Klein-Gordon field of mass $m_0$. The five-dimensional action for such a field is given by

$$S_5 = \int d^4x \int_0^{2\pi R} dy \left[ \frac{1}{2} \partial_M \Phi^* (\partial^M \Phi) - \frac{1}{2} m_0^2 \Phi^* \Phi \right] \quad (1)$$
• Inserting the mode expansion \( \Phi(x^\mu, y) \) into Eq.(1), we obtain

\[
S_5 = \frac{1}{2\pi R} \int d^4 x \int_0^{2\pi R} dy \left\{ \frac{1}{2} \sum_{mn} \partial_\mu \phi_m^* \partial^\mu \phi_n \ e^{i(n-m)y/R} 
- \frac{1}{2} \sum_{mn} \left( \frac{-im}{R} \right) \left( \frac{in}{R} \right) \phi_m^* \phi_n \ e^{i(n-m)y/R} 
- \frac{1}{2} m_0^2 \sum_{mn} \phi_m^* \phi_n \ e^{i(n-m)y/R} \right\}.
\]
• Recall that

\[ \int_0^{2\pi R} dy \ e^{i(n-m)y/R} = 2\pi R \ \delta_{mn}. \]

• Performing the y-integrations thus gives rise to the four dimensional action

\[ S_4 = \int d^4x \left\{ \frac{1}{2} \sum_n (\partial_\mu \phi_n^*)(\partial^\mu \phi_n) - \frac{1}{2} \sum_n \left[ m_0^2 + \frac{n^2}{R^2} \right] \phi_n^* \phi_n \right\}. \]

• This is the action of an infinite tower of Klein-Gordon fields \( \phi_n \), with masses given by

\[ m^2 = m_0^2 + \frac{n^2}{R^2}. \]
• If we had compactified $\delta$ extra dimensions on circles of radii $R_i$ ($i=1, 2, \ldots, \delta$) the masses is given by

$$m^2 = m_0^2 + \sum_{i=1}^{\delta} \frac{n_i^2}{R^2}. \tag{15}$$